Functional Ito calculus and applications

Rama CONT

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Abstract:

We present a new calculus for functionals of semimartingales, which extends the Ito calculus to path-dependent functionals in a non-anticipative way [1, 2, 3]. The approach builds on H. Föllmer’s deterministic proof of the Ito formula [6] and a notion of pathwise functional derivative recently proposed by B. Dupire [5]. In this framework one can derive a functional extension of the Ito formula [1], which has numerous applications in probability and mathematical finance.

The functional Ito formula is used to derive two key results. First, we obtain a martingale representation formula for square integrable functionals of a semimartingale $S$. Second, regular functionals $S$ which have the local martingale property are characterized as solutions of a functional differential equation, for which a uniqueness result is given.

These results have obvious applications to the pricing and hedging of path-dependent derivatives. We derive universal pricing and hedging equations which hold for any path-dependent option written on a financial asset whose price is modeled as a continuous semimartingale $S$. Using these results we derive a general formula for the hedging strategy of a path-dependent contingent claim and present a numerical method for computing this hedging strategy [4]. By contrast with methods based on Malliavin calculus, this representation is based on non-anticipative quantities which many be computed pathwise and leads to simple simulation-based estimators of hedge ratios [4].

These lectures are based on joint work with David FOURNIE (Columbia University).

OUTLINE

2. Functional representation of non-anticipative processes
3. Pathwise calculus for non-anticipative functionals
   b. Functions of finite quadratic variation. Föllmer’s pathwise Ito formula.
   c. A pathwise change of variable formula for functionals
4. Functional Ito calculus
   a. Functional change of variable formula for semimartingales
   b. Vertical derivative of a non-anticipative process
   c. A martingale representation formula
5. Functional Ito calculus: extension to square integrable martingales
   (a) Vertical derivative of a square integrable martingale
   (b) General martingale representation formula
   (c) Relation with Malliavin calculus
   (d) Functional characterization of local martingales.
   (e) Uniqueness of solutions to the martingale equation.

6. Applications to the pricing and hedging of path-dependent derivatives
   (a) Hedging strategies for path dependent derivatives.
   (b) Numerical computation of hedging strategies.
   (c) A universal pricing equation.
   (d) Theta-Gamma tradeoff for path-dependent options.
   (e) Example: Asian options

7. Extensions
   (a) Functionals of quadratic variation.
   (b) Example: weighted variance swaps.
   (c) Locally regular functionals and functionals involving exit times.
   (d) Local solutions of the martingale equation.
   (e) Examples: Barrier options.

References