A QUADRATIC APPROACH TO INTEREST RATES MODELS IN INCOMPLETE MARKETS

FRANCESCA BIAGINI

Abstract. The aim of this paper is to apply the mean-variance hedging approach, originally formulated for risky assets, to interest rate models in presence of stochastic volatility.
In a HJM framework, we set a finite number of bonds such the volatility matrix is invertible and provide an explicit formula for the density of the variance-optimal measure which is independent by the chosen times of maturity.
Finally, we compare the mean-variance hedging approach to the local risk minimization one in the interest rate case.

1. Introduction

The aim of this paper is to extend the mean-variance hedging approach to interest rate models in presence of stochastic volatility. The interest rate case is analysed in a Heath-Jarrow-Morton framework, where the forward rate volatility is supposed to be stochastic. Here a stochastic volatility model is seen as a model with *incomplete information*, where volatility is affected by an additional source of randomness. A perfect replication of a given european option H is not possible even by using an infinite number of bonds. In order to find an approximation price and strategy, we choose the mean-variance hedging approach and consider only self-financing portfolios composed by a finite number of bonds as in the approach of [6] .
We set $T_1 < T_2 < \cdots < T_n$ times of maturity, greater than the option time of expiration T_0, such that the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is invertible P^E-almost everywhere for every t. We characterize the set of the martingale measures for $\dfrac{p(t,T_j)}{p(t,T_0)}$, $t \leq T_0$, $j = 1, \ldots, n$ and compute an explicit formula for the density of the variance-optimal measure for $\dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1, \ldots, n$, in terms of Doleans Exponential. This expression is shown to be independent of the chosen T_j.
Finally, we introduce the local risk minimization approach for interest rates and compare it with the mean-variance hedging one.

2. The Model

In the sequel, all filtrations are supposed to satisfy the so-called *"usual hypothesis"*.
Our basic model consists of two complete filtered probability spaces denoted by $(\Omega, \mathcal{F}^W, \mathcal{F}^W_t, P^W)$ and by $(E, \mathcal{E}, \mathcal{E}_t, P^E)$. We assume that W_t is a standard n-dimensional brownian motion on $\Omega = \mathcal{C}([0,T], \mathbb{R})$, P^W is the Wiener measure and

$\mathcal{F}_t^W$ is the P^W-augmentation of the filtration generated by W_t. The space E represents an additional source of randomness which affects the market. The market is now incomplete as a result of *incomplete information*: if the evolution of η had been known the market would be complete.
We suppose that there exists on E a square integrable (eventually d-dimensional) martingale M_t endowed with the *predictable representation property*, i.e. for every square integrable martingale N_t there exists a predictable process H_t such that $N_t = N_0 + \int_0^t H_s dM_s$.
We analyse the mean-variance hedging criterion in the case of interest rates models. The assets to be considered on the market are zero coupon bonds with different maturities. As in [4], we represent the price at time t of a bond maturing at time T by an optional stochastic process $p(t,T)$ such that $p(t,t) = 1$ for all t.
We assume that there exists a frictionless market for T-bonds for every $T > 0$ and that for every fixed t, $p(t,T)$ is almost surely differentiable in the T-variable. The *forward rate* $f(t,T)$ is defined as $f(t,T) = -\dfrac{\partial \log p(t,T)}{\partial T}$ and the *short rate* as $r_t = f(t,t)$.
According to the Heath-Jarrow-Morton approach, we describe the forward rate dynamics. In this setting, $f(t,T)$ is represented by a process on the product probability space $(\Omega \times E, \mathcal{F}_t^W \otimes \mathcal{E}_t, P^W \otimes P^E)$ such that

$$df(t,T,\omega,\eta) = \alpha(t,T,\omega,\eta)dt + \sigma(t,T,\omega,\eta)dW_t(\omega) \tag{1}$$

with initial condition $f(0,T,\eta) = f^*(0,T)$. We make the following assumptions:

i) The equation (1) admits P^E-a.e. a unique strong solution with respect to the filtration $\mathcal{F}_t^W$. For example, it is sufficient that α and σ are P^E-a.e.-bounded.
ii) *Heath-Jarrow-Morton condition* on the drift: there exists a predictable $\mathbb{R}^n$-valued process h_t such that the integral $\int h_s dW_s$ is well defined and

$$\alpha(t,T,\omega,\eta) = \sigma(t,T,\omega,\eta)\int_t^T \sigma(t,s,\eta)ds - \sigma(t,T,\omega,\eta)h_t(\omega,\eta) \tag{HJM}$$

for every $T \geq 0$. For the sake of simplicity, in the sequel we will omit ω in the notation.
In the complete market case, this condition guarantees the existence of the unique equivalent martingale measure for $\dfrac{p(t,T)}{B_t}$ as long as $\mathcal{E}\left(\int hdW\right)$ is a uniformly integrable martingale, while in this setting of incomplete information there exists an infinite number of them. Note that it compels to impose stronger regularity on σ to obtain global solutions for equation (2). For a further discussion on the integrability conditions to impose on h_t, see [1].

By Proposition 15.5 of [4], we obtain the bond price dynamics:

$$\frac{dp(t,T)}{p(t,T)} = (r(t,\eta) + \frac{1}{2}\|S(t,T,\eta)\|^2 + A(t,T,\eta))dt + S(t,T,\eta)dW_t$$

where

(1) $S(t,T,\eta) = -\int_t^T \sigma(t,s,\eta)ds$
(2) $A(t,T,\eta) = -\int_t^T \alpha(t,s,\eta)ds$

Neverthless in principle an infinite number of bonds is available for trade, we consider only portfolios composed by to an arbitrarily large, but finite number of bonds as in the approach of [6]. Since are traded bonds for every time of maturity $T \in \mathbb{R}^+$, one is induced to think that the market is complete in spite of lack of information. Unfortunately, this is not true. For example, suppose in equation (1) dim $W_t = 1$ and let the volatility have a jump at a random time. The market is incomplete since the random time of jump can not be known neither through the observation of the entire term structure. For a further discussion, we refer to Example 2.1 of [1].

3. The Variance-Optimal Measure for Interest Rates

In this framework, we study the problem of hedging a certain European option H expiring at time T_0 by using a self-financing portfolio composed by a finite number of bonds of convenient maturities and eventually by the money market account B_t. In the sequel we assume to work with the filtration $(\mathcal{F}_t)_{t\in[0,T_0]}$; for the sake of simplicity we will write $\frac{dQ}{dP}$ instead of $\frac{dQ}{dP}|_{\mathcal{F}_{T_0}}$.
Since a perfect replication is not possible, we look for a self-financing portfolio which solves the following minimization problem:

$$\min E\left[(H - V_{T_0})^2\right] \tag{2}$$

Usually the money market account $B_t = \exp\left(\int_0^t r(s,\eta)ds\right)$ is used as discounting factor. Since the spot rate is now stochastic, the minimization problem (2) is equivalent to

$$\min E^B\left[(\frac{H}{B_{T_0}} - \frac{V_{T_0}}{B_{T_0}})^2\right]$$

where E^B is the expectation under the equivalent probability P^B with density

$$\frac{dP^B}{dP} = \frac{B_{T_0}^2}{E\left[B_{T_0}^2\right]}$$

The computation of the new bond dynamics under P^B can be quite complicated even in very simple cases, as shown in further details in Remark 3.9 of [1].
In order to avoid it, we can choose as numéraire the bond $p(t,T_0)$ expiring at time T_0 of maturity of H. We immediately have

$$\frac{dP^{T_0}}{dP} = \frac{p(T_0,T_0)^2}{E\left[p(T_0,T_0)^2\right]} = 1$$

or in other words $P^{T_0} \equiv P$.
More precisely, we are not simply interested in a self-financing portfolio whose final value has minimal quadratic distance by H, but, once fixed $(n+1)$ bonds $p(t,T_j)$, $j = 0,1,\dots,n$, where n is the dimension of W_t, we look for a solution to the minimization problem:

$$\min_{\substack{V_0\in\mathbb{R}\\ \theta\in\Theta}} E\left[(H - V_0 - G_{T_0}(\theta))^2\right] \tag{3}$$

where $G_t(\theta) = \int_0^t \theta_s dX_s$, $X_s^j = \dfrac{p(s,T_j)}{p(s,T_0)}$ and

$$\Theta = \left\{ \theta \in L(X) : \int \theta dX \in \mathcal{S}^2 \right\} \tag{4}$$

$L(X)$ is the set of integrable processes with respect to X_t and $\mathcal{S}^2$ is the space of square-integrable semimartingale.
We assume a sort of no-arbitrage condition on the underlying financial market:

$$\textbf{no-approximate profit condition}: \ 1 \notin G_{T_0}(\Theta) \tag{5}$$

This conditions simply means that the riskless profit 1 can't be approximate by using self-financing portfolios with zero initial wealth.
Problem (3) admits a unique solution (V_0, θ) for all $H \in L^2$ under the hypothesis that $G_{T_0}(\Theta)$ is closed (see [9] for the proof). In this case, θ is called the *mean-variance optimal* strategy and V_0 the *approximation price*. The drawback of the nonclosedness of the space $G_{T_0}(\Theta)$ can be overcome by looking for a mean-variance optimal strategy in the space Θ_{GLP} of all predictable processes such that the stochastic integral $\int_0^t \theta_s dX_s$ is a Q-square-integrable martingale for every equivalent square integrable martingale measure Q (see [9]).
Problem (3) is strictly related to a particular martingale measure for X_t, since the approximation price and the mean-variance optimal strategy θ can be computed in terms of $\tilde{P}$, the *variance-optimal measure*. We denote as $\mathcal{M}_s^2(T_1,\dots,T_n)$ and $\mathcal{M}_e^2(T_1,\dots,T_n)$ respectively the set of *signed martingale measures* and the set of *equivalent martingale measures* for $\dfrac{p(t,T_j)}{p(t,T_0)}, j = 1,\dots,n$.
The *variance-optimal* measure $\widetilde{P}$ is the element of $\mathcal{M}_s^2(T_1,\dots,T_n)$ of minimal norm, where for every $Q \in \mathcal{M}_s^2(T_1,\dots,T_n)$

$$\| \frac{dQ}{dP} \|^2 = E\left[(\frac{dQ}{dP})^2 \right]$$

If (2) has solution, in [9] it is shown that $\tilde{V}_0 = \tilde{E}[H]$. Moreover, if $G_{T_0}(\Theta)$ is closed and there exists at least a martingale measure for X_t, the optimal strategy θ can be computed by using the density of $\widetilde{P}$, as shown in [9].
Apparently, this definition of $\widetilde{P}$ depends on the chosen maturities $T_1,\dots,T_n$. By imposing the following condition, we will show in the sequel that it is actually invariant under a change of the times of maturity.

There exist maturities $T_1,\dots,T_n$ greater than T_0 such that for every t

$$\text{the matrixes } \sigma_i(t,T_j) \text{ and } \int_{T_0}^{T_j} \sigma_i(t,s)ds \text{ are non-singular } P^E\text{-a.e} \tag{H1}$$

This assumption is motivated by Proposition 4.3 of [3] and by Proposition 5.5 and Theorem 5.6 by [5]. For a further discussion, see [1].
In order to obtain an explicit formula for the variance-optimal measure, we characterize the set of the martingale measure for $\dfrac{p(t,T)}{p(t,T_0)}$ for every $T > 0$. Note that we don't need $T \le T_0$ since time t cannot exceed T_0 by assumption.

Lemma 3.1. *Let Z_t be a local martingale with $Z_0 = 1$. The following conditions are equivalent:*

(1) $Z_t \dfrac{p(t,T)}{p(t,T_0)}$ *is a local martingale for every $T > 0$*
(2) $Z_t = \mathcal{E}\left(-\int_0^\cdot (h_s + S(s,T_0,\eta))dW_s\right)_t (1 + \int_0^t k_s dM_s)$ *for some predictable process k_s such that the integral $\int_0^t k_s dM_s$ is a local martingale.*

Proof. For the proof, see Lemma 3.4 of [1]. □

Lemma 3.1 shows that our condition on the drift guarantees the existence of an absolutely continuous (eventually signed) martingale measure for $\dfrac{p(t,T)}{p(t,T_0)}$ for every $T \geq 0$, $t \leq T_0$.
Since we assume to invest in an arbitrary, but finite number of bonds, we choose for our portfolio $p(t,T_1),\dots,p(t,T_n)$ where $T_0 < T_1 < \cdots < T_n$ are maturities such that $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is invertible for P^E-almost every η. By the following lemma, we obtain that the set of martingale measures for $\dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1,\dots,n$, coincides with the set of martingales measures for $\dfrac{p(t,T)}{p(t,T_0)}$, $T \geq 0$.

Lemma 3.2. *Let Z_t be a local martingale with $Z_0 = 1$. The following conditions are equivalent:*

(1) $Z_t \dfrac{p(t,T_j)}{p(t,T_0)}$ *is a local martingale for every $j = 1,\dots,n$*
(2) $Z_t = \mathcal{E}\left(-\int_0^\cdot (h_s + S(s,T_0,\eta))dW_s\right)_t (1 + \int_0^t k_s dM_s)$ *for some predictable process k_s such that the integral $\int_0^t k_s dM_s$ is a local martingale.*

We remark that this result is independent from the chosen maturities unless for the fact that $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ must be invertible.

Proposition 3.3. (1) *If $Q \in \mathcal{M}_s^2(T_1,\dots,T_n)$, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^\cdot (h_s(\eta) + S(s,T_0,\eta))dW_s\right)_{T_0} (1 + \int_0^{T_0} k_s dM_s)$$

for some predictable process k_t such that the above expression is square integrable.

(2) *If $Q \in \mathcal{M}_e^2(T_1,\dots,T_n)$, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^\cdot (h_s(\eta) + S(s,T_0,\eta))dW_s\right)_{T_0} \mathcal{E}\left(\int_0^\cdot k_s dM_s\right)_{T_0}$$

for some predictable process k_t such that the Doleans Expontial

$$\mathcal{E}\left(-\int_0^\cdot (h_s(\eta) + S(s,T_0,\eta))dW_s + \int_0^\cdot k_s dM_s\right)_t$$

is a square-integrable martingale and $k_t \cdot \Delta M_t > -1$.

Proof. This proposition directly follows by Lemma 3.2. □

The following Lemma is quite technical, but it allows us to write an explicit expression for the density of the variance-optimal measure.

Lemma 3.4. *Let H, K be two predictable stochastic processes whose stochastic integrals $\int_0^t H_s dW_s^*$ and $\int_0^t K_s dM_s$ are defined. The following conditions are equivalent:*

$$\exp\left(\int_0^T \|(h_s(\eta)+S(s,T_0,\eta))\|^2 ds\right) = c\,\frac{\mathcal{E}\left(\int_0^\cdot H_s dW_s^*\right)_T}{\mathcal{E}\left(\int_0^\cdot K_s dM_s\right)_T} \tag{6}$$

$$\mathcal{E}\left(-\int_0^\cdot (h_s(\eta)+S(s,T_0,\eta))dW_s + \int_0^\cdot K_s dM_s\right)_T = \\ = c\,\mathcal{E}\left(\int_0^\cdot (-h_s(\eta)-S(s,T_0,\eta)+H_s)d\widehat{W}_s\right)_T \tag{7}$$

where c is the same constant in both equations.

Proof. For the proof, see Lemma 3.7 of [1]. □

We recall that $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$ and denote by A_t the matrix whose ji-th element is given by $[A_t]_{ji} = \int_{T_0}^{T_j} \sigma_i(t,s)ds$. By exploiting Lemma 3.4, we obtain the following explicit formula for the variance-optimal measure.

Theorem 3.5. *Let H, K be two predictable processes such that the exponential martingale $\mathcal{E}\left(\int_0^\cdot H_s dW_s + \int_0^\cdot K_s dM_s\right)$ is square-integrable. Then H, K are solutions of the equation (7) of Lemma 3.4 if and only if*

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(-\int_0^\cdot (h_s(\eta)+S(s,T_0,\eta))dW_s + \int_0^\cdot K_s dM_s\right)_{T_0}$$

or equivalently

$$\frac{d\widetilde{P}}{dP} = \frac{\mathcal{E}\left(-\int_0^\cdot \beta_s dX_s\right)_{T_0}}{E\left[\mathcal{E}\left(-\int_0^\cdot \beta_s dX_s\right)_{T_0}\right]}$$

where $\beta_s^j = \dfrac{p(s,T_0)}{p(s,T_j)} \sum_i (h_s^i(\eta) + S^i(s,T_0,\eta)) - H_s^i)[A_s^{-1}]_{ij}$.

In particular, if $\sigma(t,T,\eta,\omega) = \sigma(t,T,\eta)$, by [1] we obtain that the density of $\widetilde{P}$ has the form

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s\right)_{T_0} \frac{\exp\left(-\int_0^T \|\lambda_s\|^2 ds\right)}{E\left[\exp\left(-\int_0^T \|\lambda_s\|^2 ds\right)\right]} \tag{8}$$

where $\lambda_t = h_t(\eta) + S(t,T_0,\eta)$.

We stress that the characterization of $\widetilde{P}$ provided by Theorem 3.5 is independent of the chosen maturities $T_1,\ldots,T_n$ unless for the fact that matrix A_t must be invertible.

4. Examples

The Heath-Jarrow-Morton condition on the drift allows us to modelize only the forward rate volatility $\sigma(t,T,\eta)$.

Example 4.1. *First we consider the case when* $\dim W_t = 1$ *and*

$$\sigma(t,T) = \sigma_0 I_{\{t<\eta, t\leq T\}} + \sigma_1 I_{\{t\geq\eta, t\leq T\}}$$

where $\sigma_0, \sigma_1 \in \mathbb{R}^+$ *and* η *is a stopping time with a diffuse law on* $\mathbb{R}^+$. *Here we set* $E = \mathbb{R}^+$, $\mathcal{E}_t = \mathcal{B}([0,t]) \vee (t,+\infty]$ *and a fundamental martingale is given by* $M_t = I_{\{t\geq\eta\}} - a_t$, *where* a_t *is the compensator of the process* $I_{\{t\geq\eta\}}$ *associated to* η.

Example 4.2. *More generically, the volatility can be given by a Markov process in continuous time with a finite set of states* I. *By following the approach of* [8], *we choose* E *as the space of all right-continuous, left-limited functions from* $[0,\infty)$ *to* I *endowed with the filtration* $\mathcal{E}$ *generated by* η_t. *By Theorem IV.20.6 of* [8], *we obtain a set of martingales on* E *with the predictable representation property in the following way. Let* a, b *be states in* I *such that* $a \neq b$ *and define* $M_t^b = I_b(\eta_t) - I_b(\eta_0) - \int_0^t QI_b(\eta_s)ds$ *and* $H_t^a = I_a(\eta_{t-})$. *The process* $U_{ab}(t) = \int_0^t H_s^a dM_s^b$ *is a martingale by Lemma IV.21.12 of* [8] *and the family* $(U_{ab})_{a,b\in I, a\neq b}$ *has the predictable representation property.*

Example 4.3. *If the volatility is given by a multivariate point process* η_t, *there exist no finite set of martingales with the predictable representation property. By* [7], *we obtain that the compensated integer-valued random measure* $\mu - \nu$ *associated to* η_t *has the predictable representation property on* E *endowed with the smallest filtration under which* μ *is optional.*

Example 4.4. *Finally* η *can be given by a diffusion process*

$$\begin{aligned} df(t,T) &= \alpha(t,T,\eta_t)dt + \sigma(t,T,\eta_t)dW_t^1 \\ d\eta_t &= F(t,T,\eta_t)dt + G(t,T,\eta_t)dW_t^2 \end{aligned}$$

where W_t^1 *can be eventually correlated with* W_t^2.

5. A comparison with the local risk minimizing approach

An alternative approach for pricing and hedging contingent claims in incomplete markets is the local risk minimization one. The main difference with respect to mean-variance hedging is the fact that a local risk minimizing strategy perfectly replicates the value of a given option, but it is not self-financing. More precisely, suppose we want to hedge a T_0-option H. As in the previous sections, we choose $T_1 < \cdots < T_n$ satisfying (H1) holds and consider $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1,\ldots,n$. By exploiting the approach of [2], we have the following

Definition 5.1. *An* L^2-strategy *is a pair* (θ,θ^0) *such that* $\theta \in \Theta$ *and* θ^0 *is a real predictable process such that the* value process *left limit* $V_{t-} = \theta_t \cdot X_t + \theta_t^0$ *is square*

integrable for $0 \leq t \leq T_0$.
The (cumulative) cost process *is defined by* $C_t = V_t - \int_0^t \theta_s dX_s$, $0 \leq t \leq T_0$.

By Definition 5.1, we get that the portfolio's jumps coincide with the jumps in the cost process.

Definition 5.2. *Let* $H \in L^2(\mathcal{F}_{T_0}, P)$ *be a contingent claim. An* L^2*-strategy* (θ, θ^0) *with* $V_{T_0} = H \ P - a.s.$ *is called* pseudo-locally risk-minimizing *or* pseudo-optimal *for* H *if the cost process* C_t *is a* P*-martingale and is strongly orthogonal to the martingale part of* X.

By Definition 5.2 follows immediately that a contingent claim $H \in L^2(\mathcal{F}_{T_0}, P)$ admits a pseudo-optimal strategy if and only if H can be written as

$$H = H_0 + \int_0^{T_0} \xi_u dX_u + L_{T_0} \tag{9}$$

where $H_0 \in L^2(\mathcal{F}_{T_0}, P)$, $\xi \in \Theta$ and L is a square integrable martingale strongly P-orthogonal to the martingale part of X. Equation (9) is usually addressed in literature as the *Föllmer-Schweizer decomposition* of H. This is connected to a suitably chosen martingale measure, the so-called *minimal martingale measure.*

Definition 5.3. $\widehat{P}^0 \in \mathcal{M}_e^2(T_1, \ldots, T_n)$ *is the* minimal measure *(with respect to* $p(t, T_0)$ *as numéraire) if any locally square integrable local martingale which is orthogonal to the martingale part of* X *under* P *remains a local martingale under* $\widehat{P}^0$.

By Definition 5.3 follows immediately that the pseudo-optimal portfolio $\widehat{V}(\phi)$ is a local $\widehat{P}$-martingale and we get $\widehat{V}_t(\phi) = p(t, T_0)\widehat{E}^0 \left[H | \mathcal{F}_t\right]$. By Definition 5.3 and Theorem 3.5, we obtain that

$$\frac{d\widehat{P}^0}{dP} = \mathcal{E}\left(-\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s\right)_{T_0}$$

define the minimal measure's density as long as the Doleans Exponential $\mathcal{E}\left(-\int_0^{\cdot}(h_s(\eta) + S(s, T_0, \eta))dW_s\right)$ is a uniformly integrable martingale.
We compute now the pseudo-optimal strategy for a T_0-call option $C = (p(T_0, T_1) - K)^+$. The pseudo-optimal portfolio is given by $\widehat{V}_t(\phi) = p(t, T_0)\widehat{E}^0 \left[C | \mathcal{F}_t\right]$ and by exploiting the same argument as in Theorem 5.1 by [2], we obtain that the optimal strategy components are $\theta_t^0 = -K\widehat{E}^0 \left[1_A | \mathcal{F}_{t-}\right]$, $\theta_t^1 = \widehat{E}^1 \left[1_A | \mathcal{F}_{t-}\right]$ and $\theta_t^j = 0$ for all $j = 2, \ldots, n$. Note that in the local risk minimization case, the pseudo-optimal strategy depends only on two assets in spite of the dimension of the driving brownian motion. On the contrary, in [1] is shown that the mean-variance optimal strategy is based on $(n + 1)$ bonds, where $n = \dim W_t$.
We apply these results in order to compute the local risk minimizing strategy for a caplet $H = \frac{R^*}{p(T_0, T_1)}(\frac{1}{R^*} - p(T_0, T_1))^+ = \frac{R^*}{p(T_0, T_1)}K$, where K is a T_0-put option on $p(t, T_1)$. The pseudo-optimal portfolio for the caplet H is given by $\widehat{V}_t = p(t, T_1)\widehat{E}^1 \left[H | \mathcal{F}_t\right]$. For $t \leq T_0$, we have $\widehat{V}_t = p(t, T_1)\widehat{E}^1 \left[\frac{R^*}{p(T_0, T_1)}K \middle| \mathcal{F}_t\right] = p(t, T_0)\widehat{E}^0 \left[K | \mathcal{F}_t\right]$, since by [2] we have the following change of measure's formula

$\frac{d\widehat{P}^1}{d\widehat{P}^0} = p(T_0, T_1)\frac{p(0,T_0)}{p(0,T_1)}$. For $t > T_0$, $\widehat{V}_t = \widehat{E}\left[H|\mathcal{F}_t\right] = H$ since H is $\mathcal{F}_{T_0}$-measurable. Hence, the local risk-minimization strategies for the T_1-option H and for the T_0-option K coincide up to time T_0 and we can behave exactly as in the complete market case. On the contrary, in [1] is shown that mean-variance hedging strategy for H does not coincide with the one for K. The key is that in this approach we perfectly replicate the option value in spite of approximating it as in the mean-variance hedging criterium.

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Dipartimento di Matematica,, Università di Bologna,, P.zza Porta S. Donato,, 40127 Bologna, Italy

E-mail address: `biagini@dm.unibo.it`