Stochastic programs without duality gaps for objectives without a lower bound

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December 4, 2016

Abstract

This paper studies parameterized stochastic optimization problems in finite discrete time that arise in many applications in operations research and mathematical finance. We prove the existence of solutions and the absence of a duality gap under conditions that relax the boundedness assumption made by Pennanen and Perkkiö in [Stochastic programs without duality gaps, Math. Program., 136(1):91–110,2012]. We apply the result to a utility maximization problem with an unbounded utility for which we do not require the asymptotic elasticity condition.

Key words. Stochastic programming, parametric optimization, convex duality, utility maximization, asymptotic elasticity

AMS subject classifications. 52A41, 90C15, 90C31, 46A20, 91B16

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t=0}^T$ of complete sub sigma-algebras of $\mathcal{F}$ and consider the parametric dynamic stochastic optimization problem

$$\text{minimize} \quad Ef(x,u) := \int f(x(\omega), u(\omega), \omega) dP(\omega) \quad \text{over } x \in \mathcal{N}, \qquad (P_u)$$

where, for given integers n_t and m

$$\mathcal{N} = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\},$$

$u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ is the parameter and f is an extended real-valued $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$-measurable function, where $n := n_0 + \ldots + n_T$. Here and in what follows, we define the expectation of a measurable function ϕ as $+\infty$ unless the

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positive part ϕ^+ is integrable[1]. The function Ef is thus well-defined extended real-valued function on $\mathcal{N} \times L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$. We will assume throughout that the function $f(\cdot,\cdot,\omega)$ is *proper*, *lower semicontinuous* and *convex* for every $\omega \in \Omega$.

It was shown in [10] that, when applied to (P_u), the conjugate duality framework of Rockafellar [19] allows for a unified treatment of many well-known duality frameworks in operations research and mathematical finance. In that context, the absence of a duality gap is equivalent to the closedness of the optimal value function

$$\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x,u)$$

over an appropriate space of measurable functions u. Pennanen and Perkkiö [15] gave a simple algebraic condition on the integrand f that guarantees that the optimum in (P_u) is always attained and φ is closed; see Section 2 below. The condition provides a generalization of the classical no-arbitrage condition in financial mathematics (see [13, p. 749]) but it also assumes that f is bounded from below. In the financial context, the lower bound excludes, e.g., portfolio optimization problems with utility functions that are not bounded from above. That such a bound is superfluous is suggested e.g. by the results of Rásonyi and Stettner [17] where the existence of solutions to portfolio optimization in the classical perfectly liquid market was obtained for more general utility functions.

This article relaxes the boundedness assumption on f by generalizing the uniform lower bound to a conical one. It will be shown that this conical bound can be constructed in the classical setting of optimal investment for a utility function that satisfies the well-known asymptotic elasticity condition. In this sense, our condition is a far reaching extension of one the key properties of asymptotic elasticity to general stochastic optimization problems. Moreover, since a broad class of dualization schemes can be cast in to the framework of (P_u), we expect our result to have prominent applications outside mathematical finance as well. For instance, existence of solutions and the absence of the duality gap in the series of examples in [10] on constrained problems, shadow price of information, and stochastic control (stochastic problems of Bolza), can now be analyzed for unbounded objectives.

A key result for the relaxation is the simple observation formulated as Lemma 2. This extends the result on martingales differences ([5, Theorem 2]) to arbitrary process in the annihilator space of the essentially bounded adapted processes. Lemma 2 can also be utilized to characterize the dual problem associated to (P_u); this is one of the topics in a forthcoming article by Sara Biagini, Teemu Pennanen and the author [2].

In Section 2.1, we give the construction of the conical lower bound of f. The construction follows the lines of argumentation by Kramkov and Scachermayer in the proof of [9, Proposition 3.2], where the authors studied optimal investment. Here we give a variant that works for a class of stochastic optimization problems in discrete time.

[1] In particular, the sum of extended real numbers is defined as $+\infty$ if any of the terms equals $+\infty$.

We apply the main results to an optimal investment problem in liquid markets in Section 3. Under the assumptions that prices are bounded from below and that the initial prices are bounded, we recover an existence result by Rásonyi and Stettner [17, Theorem 6.2]. Moreover, here we obtain the closedness of the associated value function defined on an appropriate space of future liabilites that is important in valuation of contingent claims; see [13] and the forthcoming article [16]. We point out that asymptotic elasticity is only a necessary condition for our results on optimal investment (Theorem 10 and Corollary 11). We also provide a counterexample that some form of integrability for the price process is needed in order to have the closedness of the associated value function. This illustrates that the main result of Rásonyi and Stettner cannot be replicated if one insists on having the duality result involving an equivalent martingale measure with the usual integrability property; see Example 2 below.

In our main result, we also prove an expression for the recession function of the optimal value function in terms of f. This is of interest when, e.g., one studies robust no arbitrage and robust no scalable arbitrage properties and the related dominating markets; see [7, 14]. During the recent years, market models with proportional transaction costs and other nonlinear illiquidity effects have received an increasing interest [3, 6, 14]. Extensions of our results to such models will be analyzed in a forthcoming article [16], where we also give applications of the recession function of the optimal value function.

2 Closedness of the value function

In this section we will first establish the closedness of the value function associated with (P_u) in the space of measurable functions L^0 with respect to the convergence in measure. Using this, we will then establish the closedness on locally convex subspaces of L^0. Recall that a function is *closed* if it is lower semicontinuous and either proper or a constant. A function is *proper* if it never takes the value $-\infty$ and it is finite at some point.

An extended real-valued function h on $\mathbb{R}^n \times \Omega$, for a complete probability space Ω, is a *normal integrand* on $\mathbb{R}^n$ if h is jointly measurable and $h(\cdot, \omega)$ is lower semicontinuous for all ω; see [21, Corollary 14.34]. We assume throughout that f is an $\mathcal{F}$-measurable *proper convex normal integrand* on $\mathbb{R}^n \times \mathbb{R}^m$.

In all of the results, the statements concerning the recession functions are new. For a normal integrand, f^∞ is defined ω-wise as the *recession function* of $f(\cdot, \cdot, \omega)$, i.e.

$$f^\infty(x, u, \omega) = \sup_{\alpha > 0} \frac{f(\bar{x} + \alpha x, \bar{u} + \alpha u, \omega) - f(\bar{x}, \bar{u})}{\alpha},$$

which is independent of the choice $(\bar{x}, \bar{u}) \in \operatorname{dom} f(\cdot, \cdot, \omega)$. By [18, Theorem 8.5], the supremum equals the limit as $\alpha \to +\infty$.

Theorem 1. *Assume that there exists $m \in L^1$ such that*

$$f(x, u, \omega) \geq m(\omega)$$

for all $(x,u,\omega)\in\mathbb{R}^n\times\mathbb{R}^m\times\Omega$ *and that*

$$\mathcal{L}=\{x\in\mathcal{N}\mid f^\infty(x,0)\le 0\}$$

is a linear space. Then

$$\varphi(u)=\inf_{x\in\mathcal{N}}Ef(x,u)$$

is closed and proper in L^0 *and the infimum is attained for every* $u\in L^0$*. Moreover,*

$$\varphi^\infty(u)=\inf_{x\in\mathcal{N}}Ef^\infty(x,u).$$

Proof. Theorem 2 of [15] gives closedness of φ with respect to a locally convex topological space $\mathcal{U}\subset L^0$ but the main argument of the proof establishes closedness with respect to the convergence in measure provided that f has an integrable lower bound.

Let $\bar u\in\operatorname{dom}\varphi$ and $\bar x\in\mathcal{N}$ be such that $\varphi(\bar u)=Ef(\bar x,\bar u)$. Such $\bar x$ exists by [15, Theorem 2]. We have that

$$\varphi^\infty(u)=\sup_{\alpha>0}\frac{\varphi(\bar u+\alpha u)-\varphi(\bar u)}{\alpha}=\sup_{\alpha>0}\inf_{x\in\mathcal{N}}Ef_\alpha(x,u),$$

where

$$f_\alpha(x,u)=\frac{f(\bar x+\alpha x,\bar u+\alpha u)-f(\bar x,\bar u)}{\alpha}.$$

We have

$$\varphi^\infty(u)\le\inf_{x\in\mathcal{N}}\sup_{\alpha>0}Ef_\alpha(x,u)\le\inf_{x\in\mathcal{N}}E[\sup_{\alpha>0}f_\alpha(x,u)]=\inf_{x\in\mathcal{N}}Ef^\infty(x,u).$$

To prove the converse, let $a>\sup_{\alpha>0}\inf_{x\in\mathcal{N}}Ef_\alpha(x,u)$. For every positive integer α, there is an $x^\alpha\in\mathcal{N}$ with $Ef_\alpha(x^\alpha,u)<a$. The functions f_α are nondecreasing in α, so $Ef_1(x^\alpha,u)<a$ and we may proceed as in the proof of [15, Theorem 2] (where we may assume that $x^\alpha_t\in N_t^\perp$ for every t by [15, Lemma 2]) to obtain a sequence of convex combinations $\tilde x^\alpha=\operatorname{co}\{x^{\alpha'}\mid\alpha'\ge\alpha\}$ such that $\tilde x^\alpha\to\bar x$ almost surely. By Fatou's Lemma,

$$Ef^\infty(\bar x,u)\le a,$$

which completes the proof. □

When extending the theorem above to objectives that do not have a uniform lower bound, a key role is played by the following lemma, where

$$\mathcal{N}^\perp=\{v\in L^1(\Omega,\mathcal{F},\mathbb{R}^n)\,|\,E(x\cdot v)=0\ \forall x\in\mathcal{N}^\infty\}$$

is the *annihilator* of $\mathcal{N}^\infty=L^\infty\cap\mathcal{N}$. For an extended real-valued function ϕ, we denote $\phi^+=\max\{\phi,0\}$ and $\phi^-=\max\{-\phi,0\}$.

Lemma 2. *Let* $x\in\mathcal{N}$ *and* $v\in\mathcal{N}^\perp$*. If* $E[x\cdot v]^+<\infty$*, then* $E(x\cdot v)=0$.

Proof. Any $\xi \in L^1$ can be represented as $\xi = \sum_{t'=0}^{T+1} \Delta\xi_{t'}$ where $\xi_{-1} = 0$ and $(\xi_{t'})_{t'=0}^{T+1}$ is the martingale defined as $\xi_{T+1} = \xi$ and $\xi_{t'} = E_{t'}\xi$. For $\xi^i = v_i$, we have $\xi^i_{t'} = 0$ for all $t' \leq i$, so $x_t \cdot v_t = m^i_{T'+1}$, where m^i is a local martingale defined by

$$m^i_t = \sum_{t'=0}^{t} x_i \cdot \Delta\xi^i_{t'}.$$

Thus $x \cdot v = m_{T+1}$ for $m = \sum_{i=0}^{T} m^i$. Since $Em_{T+1} < \infty$, we have that m is a martingale ([5, Theorem 2]) and thus $E(x \cdot v) = Em_{T+1} = Em_0 = 0$. □

Remark 1. To prove the above lemma, one cannot directly argue by first approximating x by $x^\nu_t = \mathbb{1}_{|x_t|\leq\nu} x_t$ for each t, and then concluding by the monotone or the dominated convergence theorem. Indeed, assume that $n_0 = n_1 = 1$, $T = 1$, and that there exists a nonnegative $\xi = L^0(\Omega, \mathcal{F}_0, P; \mathbb{R})$ and $v \in \mathcal{N}^\perp$ with $v_0 = v_1/2$ such that ξv_1^+ is not integrable and that $P(\xi \in A) > 0$ for any nonnegative nonempty open interval A. Then, for $x_0 = -2\xi$ and $x_1 = \xi$, we have that $x \cdot v = 0$, but

$$[x^\nu \cdot v]^+ = \xi v_1^+(\mathbb{1}_{\{\xi\leq\nu\}} - \mathbb{1}_{\xi\leq\nu/2}),$$

so $([x^\nu \cdot v]^+)_{\nu=1}^\infty$ is not a monotone sequence, and the smallest dominant for this sequence is the nonintegrable $\xi v_1^+ \mathbb{1}_{\{\xi\geq 1/2\}}$.

The following additivity property of the extended real-valued expectation will often be used without a mention.

Lemma 3. *Let ϕ_1 and ϕ_2 be extended real-valued measurable functions. If either $\phi_1^+, \phi_2^+ \in L^1$ or $\phi_2 \in L^1$, then*

$$E(\phi_1 + \phi_2) = E\phi_1 + E\phi_2.$$

The main contribution of this paper is contained in the following result which relaxes the lower bound of f with respect to x.

Theorem 4. *Assume that there exist $\lambda > 0$ and $(v, \beta) \in \mathcal{N}^\perp \times L^1$ such that*

$$f(x, u, \omega) \geq x \cdot v(\omega) + \lambda[x \cdot v(\omega)]^+ + \beta(\omega)$$

for all $(x, u, \omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$ and that

$$\mathcal{L} = \{x \in \mathcal{N} \mid f^\infty(x, 0) \leq 0\}$$

is a linear space. Then

$$\varphi(u) = \inf_{x\in\mathcal{N}} Ef(x, u)$$

is closed and proper in L^0 and the infimum is attained for every $u \in L^0$. Moreover,

$$\varphi^\infty(u) = \inf_{x\in\mathcal{N}} Ef^\infty(x, u).$$

Proof. Note first that $\mathcal{L} = \{x \in \mathcal{N} \mid f^\infty(x,0) - x \cdot v \leq 0\}$. Indeed, the lower bound on f implies $f^\infty(x,0,\omega) \geq x \cdot v(\omega) + \lambda[x \cdot v(\omega)]^+$, so if x belongs either to $\mathcal{L}$ or $\{x \in \mathcal{N} \mid f^\infty(x,0) - x \cdot v \leq 0\}$, then $[x \cdot v]^+ \leq 0$ and thus, $x \cdot v = 0$, by Lemma 2. Applying Theorem 1 to the normal integrand $f_v(x,u,\omega) := f(x,u,\omega) - x \cdot v(\omega)$ we see that

$$\varphi_v(u) = \inf_{x \in \mathcal{N}} Ef_v(x,u)$$

is closed and proper, the infimum is attained for every $u \in L^0$ and that

$$\varphi_v^\infty(u) = \inf_{x \in \mathcal{N}} Ef_v^\infty(x,u).$$

If either $(x,u) \in \operatorname{dom} Ef$ or $(x,u) \in \operatorname{dom} Ef_v$, the lower bound implies $[x \cdot v]^+ \in L^1$ so that $x \cdot v \in L^1$ and $E(x \cdot v) = 0$, by Lemma 2. In both cases, Lemma 3 gives

$$Ef_v(x,u) = Ef(x,u) - E(x \cdot v) = Ef(x,u)$$

so that $\varphi_v = \varphi$. Similarly, $Ef^\infty = Ef_v^\infty$ so that $\varphi^\infty(u) = \inf_{x \in \mathcal{N}} Ef^\infty(x,u)$. □

Theorem 4 is not valid if, in the lower bound for f, we allow λ to be zero.

Example 1. *Let $n = 1$ and let $\mathcal{F}_0$ be such that there exist $\alpha \in L^1(\mathcal{F}_0)$ with $\alpha \notin L^2$ and nonzero $\beta \in L^\infty$ independent of $\mathcal{F}_0$ with $E[\beta \mid \mathcal{F}_0] = 0$. We define*

$$f(x,u,\omega) = \|x - \alpha(\omega)\| + x \cdot v(\omega),$$

where $v = \alpha\beta$ so that $v \in \mathcal{N}^\perp$ and $f(x) \geq x \cdot v$. The reader may verify that

$$\inf_{x \in \mathcal{N}} Ef(x,u) = 0$$

and the only nontrivial candidate for $Ef(x,u) = 0$ is $x = \alpha$, but $Ef(\alpha,u) = \infty$.

The rest of this section is concerned with closedness of the value function on locally convex subspaces of L^0. We will assume from now on that the parameter u belongs to a decomposable space $\mathcal{U} \subset L^0$ which is in separating duality with another decomposable space $\mathcal{Y} \subset L^0$ under the bilinear form

$$\langle u, y \rangle = E(u \cdot y).$$

Recall that $\mathcal{U}$ is *decomposable* if

$$\mathbb{1}_A u + \mathbb{1}_{\Omega \setminus A} u' \in \mathcal{U}$$

whenever $A \in \mathcal{F}$, $u \in \mathcal{U}$ and $u' \in L^\infty$; see e.g. [20].

The families of closed convex sets coincide in the weak $\sigma(\mathcal{U},\mathcal{Y})$ and in the Mackey topologies $\tau(\mathcal{U},\mathcal{Y})$ [23, p. 132], so we may say that φ is closed in $\mathcal{U}$ whenever it is so with respect to either of the topologies. The closed convex function

$$\varphi^*(y) = \sup_{u \in \mathcal{U}} \{\langle u, y \rangle - \varphi(u)\}$$

is called the *conjugate* of φ. When φ is closed, it has the *dual representation*

$$\varphi(u) = \sup_{y\in\mathcal{Y}}\{\langle u,y\rangle - \varphi^*(y)\};$$

see [19, Theorem 5]. Dual representations are behind many fundamental results in mathematical finance as in the classical formula where superhedging prices of contingent claims in liquid markets are given in terms of martingale measures; for this and some recent developments in more general market models, see [11] and [12].

Using topological arguments on decomposable spaces (see e.g. Rockafellar [20] or Ioffe [4]), we may relax in Theorem 5 the lower bound on f with respect to u as well. The following theorem generalizes [15, Theorem 2] by relaxing the uniform lower boundedness of f with respect to x.

Theorem 5. *Assume that there exist $\lambda > 0$, $v \in \mathcal{N}^\perp$ and a convex normal integrand g on $\mathbb{R}^m \times \Omega$ such that Eg is $\tau(\mathcal{U},\mathcal{Y})$-continuous, Eg^∞ is finite on $\mathcal{U}$,*

$$f(x,u,\omega) \geq x\cdot v(\omega) + \lambda[x\cdot v(\omega)]^+ - g(u,\omega)$$

for all $(x,u,\omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$ and such that

$$\mathcal{L} = \{x \in \mathcal{N} \mid f^\infty(x,0) \leq 0\}$$

is a linear space. Then

$$\varphi(u) = \inf_{x\in\mathcal{N}} Ef(x,u)$$

is closed and proper in $\mathcal{U}$ and the infimum is attained for every $u \in \mathcal{U}$. Moreover,

$$\varphi^\infty(u) = \inf_{x\in\mathcal{N}} Ef^\infty(x,u).$$

Proof. Applying Theorem 4 to the normal integrand $f_g(x,u,\omega) = f(x,u,\omega) + g(u,\omega)$, we get that

$$\varphi_g(u) = \inf_{x\in\mathcal{N}} Ef_g(x,u)$$

is proper and closed on L^0, that the infimum is attained for every $u \in L^0$ and that

$$\varphi_g^\infty(u) = \inf_{x\in\mathcal{N}} Ef_g^\infty(x,u).$$

Closedness in L^0 implies closedness in the relative topology of L^1 which, by [15, Lemma 6], implies that φ_g is closed in $\mathcal{U}$. By Lemma 3, $E[f(x,u) + g(u)] = Ef(x,u) + Eg(u)$ for all $(x,u) \in \mathcal{N}\times\mathcal{U}$ and thus

$$\varphi_g = \varphi + Eg$$

on $\mathcal{U}$. The $\tau(\mathcal{U},\mathcal{Y})$-continuity of Eg implies the $\tau(\mathcal{U},\mathcal{Y})$-closedness of $\varphi = \varphi_g - Eg$.

As to the recession function, $f_g^\infty = f^\infty + g^\infty$ [18, Theorem 9.3], so

$$\varphi_g^\infty(u) = \inf_{x\in\mathcal{N}} Ef^\infty(x,u) + Eg^\infty(u).$$

Similarly, closedness of φ and Eg imply

$$\varphi_g^\infty = \varphi^\infty + (Eg)^\infty,$$

where, by the monotone convergence theorem, $(Eg)^\infty = Eg^\infty$. □

The following lemma gives a sufficient condition for the first hypothesis in Theorem 5. The characterization involves the conjugate normal integrand $f^* : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ of f defined ω-wise as follows

$$f^*(v, y, \omega) = \sup\{x \cdot v + u \cdot y - f(x, u, \omega)\}.$$

By [21, Theorem 14.50], f^* is indeed a normal integrand.

Lemma 6. *Assume that there exists $v \in \mathcal{N}^\perp$ and $\lambda > 0$ such that the function*

$$\gamma(v) = \inf_{y \in \mathcal{Y}} Ef^*(v, y)$$

is finite at v and $(1 + \lambda)v$. Then

$$f(x, u, \omega) \geq x \cdot v(\omega) + \lambda[x \cdot v(\omega)]^+ - g(u, \omega)$$

for all $(x, u, \omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$, where $g(u, \omega) = \max_i u \cdot y^i(\omega) + \beta(\omega)$ for some $y^i \in \mathcal{Y}$ and $\beta \in L^1$.

Proof. The assumption means that there exist $y^1, y^2 \in \mathcal{Y}$ and $\beta^1, \beta^2 \in L^1$ such that

$$f^*(\lambda^i v, y^i) \leq \beta^i$$

where $\lambda^1 = \lambda$ and $\lambda^2 = 1 + \lambda$. Equivalently,

$$f(x, u, \omega) \geq \lambda^i x \cdot v(\omega) + u \cdot y^i(\omega) - \beta^i(\omega) \quad i = 1, 2$$

for all $(x, u, \omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$, which implies

$$f(x, u, \omega) \geq \max_i\{\lambda^i x \cdot v(\omega)\} + \min_i u \cdot y^i(\omega) - \min_i \beta^i(\omega)$$

for all $(x, u, \omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$. □

2.1 The existence of the lower bound

In this section we construct a lower bound $v \in \mathcal{N}^\perp$ appearing in Lemma 6. Much like in the proof of [9, Proposition 3.2], the construction is based on the assumption that there are approximating problems for which the lower bounds certainly exist. We say that an integrand h is *nonincreasing towards* $\operatorname{dom} h$ if

$$h(v + v', \omega) \leq h(v, \omega) \quad \forall v, v' \in \operatorname{dom} h(\cdot, \omega) \quad P\text{-a.s.}$$

Assumption 1. There exists $\lambda \in (0,1)$ such that

$$\lambda \operatorname{dom} Ef^* \subseteq \operatorname{dom} Ef^*,$$

and there is a nonincreasing sequence $(f_n)_{n=1}^{\infty}$ of convex normal integrands with

$$f(x,u,\omega) = \inf_n f_n(x,u,\omega),$$

each f_n is bounded from below, $\Delta f_n^* := f_n^* - f_{n-1}^*$ are bounded and nonincreasing towards $\operatorname{dom} f_n^*$, and $\operatorname{dom} Ef_0 \cap L^\infty \neq \emptyset$.

A principal example of such approximating sequence can be obtained using Lemma 9 below. This lemma is geared towards utility maximization problems studied in the next section where we also show how the scaling condition in Assumption 1 is satisfied under asymptotic elasticity condition of the utility function.

To formulate the main theorem of this section, we define a mapping A from $L^\infty(\Omega;\mathbb{R}^n)$ to $L^\infty(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)$ by

$$Ax = ({}^a x, 0),$$

where $({}^a x)_t = E_t x_t$ is the *adapted projection* of x. The algebraic interior (core) in the theorem is taken with respect to the norm-topology of $L^\infty(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)$. The core-condition would allow us to utilize Rockafellar-Fenchel type duality scheme in the setting where L^∞ is paired with its Banach dual of finitely additive measures. Here our aim is to construct a feasible dual variable in L^1, so, instead, we merely use the core-condition to get boundedness in L^1. In the next section, we will show that this condition cannot be omitted in the context of optimal investment.

Theorem 7. *In addition to Assumption 1, assume that*

$$\mathcal{L} = \{x \in \mathcal{N} \mid f^\infty(x,0) \le 0\}$$

is a linear space, $f_n^\infty = f^\infty$ for each n, and that φ is finite for some $\bar{u}$ with

$$(0,\bar{u}) \in \operatorname{core}(\operatorname{dom} Ef_0 - \operatorname{rge} A).$$

Then there exists $(v,y) \in \mathcal{N}^\perp \times L^1$ such that, for all $\lambda' \in (0,1]$,

$$Ef^*(\lambda' v, \lambda' y) < \infty.$$

In particular, φ is $\sigma(L^\infty, L^1)$-closed.

Proof. We remark first that, by Lemma 6 and Theorem 5, φ is $\sigma(L^\infty, L^1)$-closed as soon as we prove the first claim. To have this, we may assume, by translation, that $\bar{u} = 0$. We define

$$\tilde{\varphi}_n(z,u) := \inf_{x \in \mathcal{N}^\infty} Ef_n(x+z,u).$$

By Theorem 5, each $\tilde\varphi_n$ is $\sigma(L^\infty,L^1)$-closed. Since $\varphi(0)\le\tilde\varphi_n(0,0)=(\operatorname{cl}\tilde\varphi_n)(0,0)$, there exists $(v^n,y^n)\in L^1\times L^1$ such that

$$\tilde\varphi_n^*(v^n,y^n)-1/n\le-\varphi(0).$$

By the interchange rule [21, Theorem 14.60], we have

$$\tilde\varphi_n^*(v,y)=Ef_n^*(v,y)+\delta_{\mathcal{N}^\perp}(v)$$

for every $(v,y)\in L^1\times L^1$. Thus, by Lemma 8 below, it suffices to show that φ_0^* has bounded level-sets in $L^1\times L^1$.

Since $\tilde\varphi_0$ can be expressed as

$$\tilde\varphi_0(z,u)=\inf_{x\in L^\infty}Ef_0(Ax+(z,u)),$$

the core-condition implies that $\tilde\varphi_0$ is bounded from above in a norm neighborhood of the origin; see [19, Theorem 18 and Example 11']. By [19, Theorem 10], $\tilde\varphi_0^*$ has compact level-sets in the Banach dual $(L^\infty\times L^\infty)^*$, so its level-sets restricted to $L^1\times L^1$ are bounded. □

The following lemma was used in the proof of Theorem 7.

Lemma 8. *In addition to Assumption 1, let*

$$\sup_n Ef_n^*(v^n,y^n)<\infty$$

for some L^1-bounded sequence $(v^n,y^n)_{n=1}^\infty$ in $\mathcal{N}^\perp\times L^1$. Then there exists $(v,y)\in\mathcal{N}^\perp\times L^1$ such that, for all $\lambda'\in(0,1]$,

$$Ef^*(\lambda'v,\lambda'y)<\infty.$$

Proof. Komlos' theorem (see e.g. [8, Theorem 5.2.1]) implies that there is a sequence of convex combinations $(\tilde v^n,\tilde y^n)\in\operatorname{co}\{(v^{n'},y^{n'})\mid n'\ge n\}$ such that $(\tilde v^n,\tilde y^n)$ converge to some $(\bar v,\bar y)\in L^1\times L^1$ almost surely. By the assumption that $\sup_n Ef_n^*(v^n,y^n)$ is finite, there is an $\alpha\in\mathbb{R}$ such that

$$Ef_n^*(v^n,y^n)-1/(n+1)\le\alpha.$$

Since $\operatorname{dom}Ef_0\cap L^\infty\ne\emptyset$, we get from Fatou's lemma and convexity that

$$E[f^*(\bar v,\bar y)]\le\liminf_n Ef_n^*(\tilde v^n,\tilde y^n)\le\alpha. \tag{1}$$

Let $\lambda^\nu>0$ be such that $\sum_{\nu=1}^\infty\lambda^\nu=1$. By the scaling condition in Assumption 1, we have $Ef^*(\lambda^\nu\bar v,\lambda^\nu\bar v)<\infty$. Since f_n^* increase to f^*, there is, for every ν, an $n(\nu)$ such that

$$Ef_n^*(\lambda^\nu\bar v,\lambda^\nu\bar y)\ge Ef^*(\lambda^\nu\bar v,\lambda^\nu\bar y)-\lambda^\nu \tag{2}$$

for all $n \geq n(\nu)$. Each of the functions $\Delta f^*_\nu := f^*_{n(\nu+1)} - f^*_{n(\nu)}$ is assumed to be bounded, so, by a diagonalization argument, we may assume that

$$E\Delta f^*_\nu(\lambda^\nu \tilde{v}^n, \lambda^\nu \tilde{y}^n) \leq E\Delta f^*_\nu(\lambda^\nu \bar{v}, \lambda^\nu \bar{y}) + \lambda^\nu$$

for all $n \geq n(\nu)$. Since $\Delta f^*_\nu \leq f^* - f^*_{n(\nu)}$ and since we have (2), we get that

$$E\Delta f^*_\nu(\lambda^\nu \tilde{v}^n, \lambda^\nu \tilde{y}^n)) \leq \lambda^\nu \tag{3}$$

for all $n \geq n(\nu)$. We define

$$(\tilde{v}, \tilde{y}) = \sum_{\nu=0}^{\infty} \lambda^\nu (\tilde{v}^\nu, \tilde{y}^\nu).$$

It is easy to verify that closed subspaces of Banach spaces are closed under countable convex combinations of bounded sequences, so $(\tilde{v}, \tilde{y}) \in \mathcal{N}^\perp \times L^1$. Thus it suffices to show that $(\tilde{v}, \tilde{y}) \in \operatorname{dom} Ef^*$. Firstly, since f^*_n increase to f^*, we see from (1) that $(f^*_{n(0)}(\tilde{v}^\nu, \tilde{y}^\nu))_{\nu=1}^\infty$ is bounded in L^1 and thus $Ef^*_{n(0)}(\tilde{v}, \tilde{y}) \leq \sum_{\nu=0}^\infty \lambda^\nu Ef^*_{n(0)}(\tilde{v}^\nu, \tilde{y}^\nu) < \infty$. Secondly Δf^*_ν are are assumed to be nonincreasing towards $\operatorname{dom} f^*_\nu$, so

$$E\Delta f^*_\nu(\tilde{v}, \tilde{y}) \leq E\Delta f^*_\nu((\lambda^\nu \tilde{v}^\nu, \lambda^\nu \tilde{y}^\nu)) \leq \lambda^\nu.$$

Thus

$$Ef^*(\tilde{v}, \tilde{y}) = Ef^*_{n(0)}(\tilde{v}, \tilde{y}) + \sum_{\nu=0}^{\infty} E\Delta f^*_\nu(\tilde{v}, \tilde{y}) < \infty.$$

□

The following lemma gives an approximating sequence satisfying Assumption 1.

Lemma 9. *Let g be a proper closed convex monotone function on the real line and $g_n = \max\{g, -n\}$. Then each $g^*_{n-1} - g^*_n$ is bounded and nonincreasing towards* $\operatorname{dom} g^*_n$.

Proof. By symmetry, we may assume that g is nondecreasing. Since $g_n \geq g_{n-1} + 1$, we have $\Delta g^*_n \leq 1$ on $\operatorname{dom} g^*_n$, so Δg^*_n are bounded on $\operatorname{dom} g^*_n$.

Recall that, for any convex function g, $y \in \partial g(x)$ is equivalent to $x \in \partial g^*(y)$, where ∂g denotes the subdifferential mapping, that is, $y \in \partial g(x)$ if $g(x') \geq g(x) + \langle x - x', y\rangle$ for all x'. Since here g is nondecreasing and $g_n = \max\{g, -n\}$, we have, for any $y, x, x' \in \mathbb{R}$, that if $y \in \partial g_n(x) \cap \partial g_{n-1}(x')$, then $x \leq x'$. Thus

$$\partial g^*_n(y) \subseteq \partial g^*_{n-1}(y) + \mathbb{R}_-,$$

so we see that Δg^*_n are nonincreasing on $\operatorname{dom} g^*_n$.

□

3 Application to mathematical finance

We will consider optimal investment on a financial market with a finite set J of assets from the point of view of an agent who has a financial liability described by a payment $u \in L^0(\mathcal{F}_T)$ to be made at the terminal time. As is usual, we express the terminal wealth as a stochastic integral with respect to the adapted price process s so that the problem can be written as

$$\text{minimize} \quad EV\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right) \quad \text{over} \quad x \in \mathcal{N}_0, \tag{ALM}$$

where $\mathcal{N}_0 = \{x \in \mathcal{N} \mid x_T = 0 \ P\text{-a.s.}\}$, $x_{-1} = 0$ and V is a nondecreasing nonconstant convex function with $V(0) = 0$. The objective of the agent is thus to find a trading strategy x that hedges against the liability u as well as possible as measured by expected "disutility" at terminal time. The change of signs $U(u) = -V(-u)$ transforms the problem into a usual maximization of expected terminal utility. There is vast literature on (ALM) and on its extensions to more sophisticated models of optimal investment; see the references in [13].

The following theorem gives conditions for the closedness in $\mathcal{U}$ of the value function φ of (ALM) and thus, the validity of the dual representation

$$\varphi(u) = \sup_{y \in \mathcal{Y}} \{\langle u, y\rangle - \varphi^*(y)\}.$$

Recall that the market satisfies the *no-arbitrage condition* if

$$\left\{\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \;\middle|\; \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq 0,\ x \in \mathcal{N}\right\} = \{0\}. \tag{NA}$$

Theorem 10. *Assume that the market satisfies* (NA) *and that there exists a martingale measure $Q \ll P$ of s such that $EV^*(\lambda \frac{dQ}{dP}) < \infty$ for two different $\lambda \geq 0$. If $\frac{dQ}{dP} \in \mathcal{Y}$, then the value function of* (ALM) *is closed in $\mathcal{U}$ and* (ALM) *has a solution for all $u \in \mathcal{U}$.*

Proof. We fit (ALM) in the general model (P_u) with[2]

$$f(x, u, \omega) = V\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_t(\omega)\right) + \delta_0(x_T).$$

We have

$$f^\infty(x, 0, \omega) = V^\infty\left(-\sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right) + \delta_0(x_T).$$

[2] Here δ_C denotes the *indicator function* of a set C, i.e. $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise.

Since V is nonconstant, we get $V^\infty(u) > 0$ for $u > 0$ and $V^\infty(u) \leq 0$ for $u \leq 0$, so the linearity condition in Theorem 5 reduces to (NA). In order to apply Lemma 6, we calculate

$$\begin{aligned}
f^*(v,y,\omega) &= \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}} \left\{ x\cdot v + uy - V\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega)\right) - \delta_0(x_T)\right\} \\
&= V^*(y) + \sup_{x\in\mathbb{R}^n} \left\{ \sum_{t=0}^{T-1} x_t\cdot(y\Delta s_{t+1}(\omega) + v_t)\right\} \\
&= V^*(y) + \sum_{t=0}^{T-1} \delta_0(y\Delta s_{t+1}(\omega) + v_t).
\end{aligned}$$

We choose $y = \frac{dQ}{dP}$ and $v = -\frac{dQ}{dP}\Delta s_{t+1}$ so that

$$Ef^*(\lambda v, \lambda y) = EV^*\left(\lambda \frac{dQ}{dP}\right).$$

Here $Ef^*(\lambda v, \lambda y)$ is finite for two different λ by assumption so that, by Lemma 6, we may apply Theorem 5. □

Remark 2. The proof of Theorem 10 also gives a formula for the recession function of the value function associated with (ALM). That is, since the value function is closed, its recession function is closed as well which, by the recession formula in Theorem 5, is equivalent with the fact that the set

$$\mathcal{C} = \{u \in \mathcal{U} \,|\, \exists x \in \mathcal{N}_0 : \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \geq u\}$$

of claims that can be superhedged without a cost is closed in $\mathcal{U}$. However, this result follows already from [15, Theorem 2].

Remark 3. Under the assumptions of Theorem 10, if one is merely interested in the existence of solutions of (ALM), we point out the following. We may define

$$f(x,u,\omega) = V\left(c(\omega) - \sum_{t=0}^{T-1} x_t\cdot\Delta s_{t+1}(\omega)\right) + \delta_0(x_T)$$

so that f is independent of u and the Fenchel inequality implies that

$$f(x,u,\omega) \geq x\cdot v(\omega) + c(\omega)\lambda\frac{dQ}{dP}(\omega) - V^*(\lambda\frac{dQ}{dP}(\omega))$$

for all $(x,u,\omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \Omega$, where $v \in \mathcal{N}^\perp$ is defined by $v_t = -\lambda\frac{dQ}{dP}\Delta s_{t+1}$. Thus, by Theorem 4, (ALM) has a solution for $u = c$ whenever c is Q-integrable.

Remark 4. In general, the price process does not admit a martingale measure for which the integrability condition in Theorem 10 is satisfied. For example, consider a one-period market model with a trivial σ-algebra $\mathcal{F}_0$, $P(\Delta s_1 = 1) = \frac{3}{4}$, $P(\Delta s_t = -1) = \frac{1}{4}$, and a disutility function specified by

$$V'(u) = \sum_{n=1}^{\infty} \left[\mathbb{1}_{(-n-1,-n]}(u)(1 + \frac{1}{n^2}) + \mathbb{1}_{(n,n+1]}(u)(3 - \frac{1}{n^2}) \right].$$

This is Example 7.3 in [17] where it has been shown that the optimal value of (ALM) for $u = 0$ is finite but optimal solutions do not exist. The interested reader may verify that the unique martingale measure is given by $\frac{dQ}{dP} = \frac{2}{3}(\mathbb{1}_{\Delta s_1 = 1} + 3\mathbb{1}_{\Delta s_1 = -1})$ and that $V^*(y) = +\infty$ whenever $y \notin [1, 3]$ so that $EV^*(\lambda \frac{dQ}{dP}) = +\infty$ for every $\lambda \neq \frac{3}{2}$.

The "two-λ-condition" in the theorem is close in spirit to [1, Assumption 4.2] since it implies a similar λ-condition for all λ sufficiently small [1, Corollary 4.4]. In the cited article the authors work in a continuous time setting, here two different λ suffice.

The following corollary says that the two-λ-condition is implied by a simpler integrability condition for utility functions that satisfy either of the well-known *asymptotic elasticity* conditions

$$V^*(\lambda y) \leq CV^*(y) \quad \forall y \in [0, \bar{y}] \text{ for some } C < +\infty,\ \bar{y} > 0,\ \lambda \in (0, 1), \tag{4}$$

$$V^*(\lambda y) \leq CV^*(y) \quad \forall y \geq \bar{y} \text{ for some } C < +\infty,\ \bar{y} > 0,\ \lambda > 1. \tag{5}$$

These conditions together with their equivalent formulations were introduced in [9] and [22], respectively.

Corollary 11. *Assume that the market satisfies* (NA)*,* V *satisfies* (4) *or* (5)*, and that there exists a martingale measure* $Q \ll P$ *of* s *such that* $EV^*(\lambda \frac{dQ}{dP}) < \infty$ *for some* $\lambda \in \mathbb{R}_+$*. If* $\frac{dQ}{dP} \in \mathcal{Y}$*, then the value function of* (ALM) *is closed in* $\mathcal{U}$ *and* (ALM) *has a solution for all* $u \in \mathcal{U}$*.*

Proof. We use the notation from the proof of Theorem 10. For simplicity, we assume that $\lambda = 1$. If we have (4), then

$$\begin{aligned} EV^*(\lambda' \frac{dQ}{dP}) &= E\mathbb{1}_{\{\frac{dQ}{dP} \leq \bar{y}\}} V^*(\lambda' \frac{dQ}{dP}) + E\mathbb{1}_{\{\frac{dQ}{dP} > \bar{y}\}} V^*(\lambda' \frac{dQ}{dP}) \\ &\leq E\mathbb{1}_{\{\frac{dQ}{dP} \leq \bar{y}\}} V^*(\lambda' \frac{dQ}{dP}) + E\mathbb{1}_{\{\frac{dQ}{dP} > \bar{y}\}} \max\{V^*(\frac{dQ}{dP}), V^*(\lambda' \bar{y})\} \\ &\leq E\mathbb{1}_{\{\frac{dQ}{dP} \leq \bar{y}\}} CV^*(\frac{dQ}{dP}) + E\mathbb{1}_{\{\frac{dQ}{dP} > \bar{y}\}} \max\{V^*(\frac{dQ}{dP}), CV^*(\bar{y})\}, \end{aligned}$$

so $Ef^*(\lambda' v, \lambda' y)$ is finite for some $\lambda' < 1$. Similary, if we have (5), then $Ef^*(\lambda' v, \lambda' y)$ is finite for some $\lambda' > 1$. □

The next theorem gives conditions for the existence of a martingale measure used in Corollary 11. The third condition in the theorem means that the optimal value of (ALM) is finite for some positive initial endowment. We say that the price process is bounded from below if there is a constant $a \leq 0$ such that

$s_t^j \geq a$ P-almost surely for all t and $j \in J$. In the proof, much like in that of [9, Proposition 3.2], we approximate the disutility function by disutility functions that are bounded from below. In the theorem, we assume the price process to be bounded towards the negative orthant, but of course any orthant would do.

Theorem 12. *Assume that the market satisfies* (NA)*, V satisfies* (4)*, the optimal value of* (ALM) *is finite for some $u = c$, where $c < 0$ is a constant, the price process is bounded from below, and that $s_0 \in L^\infty$. Then there exists a martingale measure $Q \ll P$ of s with $EV^*(\lambda \frac{dQ}{dP}) < \infty$ for some $\lambda \in \mathbb{R}_+$.*

Proof. Our aim is to apply Theorem 7. Define $V_n = \max\{V, -n\}$

$$f(x, u, \omega) = V_n\left(u - \sum_{t=0}^{T-1} x_t \Delta s_{t+1}(\omega)\right),$$
$$f_n(x, u, \omega) = V_n\left(u - \sum_{t=0}^{T-1} x_t \Delta s_{t+1}(\omega)\right)$$

so that Assumption 1 is satisfied. Indeed, the scaling condition follows as in the proof of Corollary 11, and since

$$f_n^*(v, y, \omega) = V_n^*(y) + \sum_{t=0}^{T-1} \delta_{\{0\}}(y \Delta s_{t+1}(\omega) + v_t),$$

the approximating sequence satisfies the required assumptions by Lemma 9. To prove the core-condition of Theorem 7, let $(z, u) \in L^\infty \times L^\infty$. Since s is bounded from below and $s_0 \in L^\infty$, there is a constant $M > 0$ such that, by choosing $x_0^z = 2T\|z\|$ and $\Delta x_t^z = -2\|z\|$ for $t = 1, \ldots, T$, we have

$$-\sum_{t=0}^{T-1}(x_t^z + z_t) \cdot \Delta s_{t+1} = (x_0^z + z_0) \cdot s_0 + \sum_{t=0}^{T} s_t \cdot (\Delta x_t^z + \Delta z_t) \leq M\|\|z\|.$$

For λ small enough, we see that $Ef_0(\lambda(x^z + z), \bar{u} + \lambda u) < \infty$ so that the core-condition is satisfied. Thus, by Theorem 7, there exists $(v, y) \in \mathcal{N}^\perp \times L^1$ such that $Ef^*(v, y) < \infty$. Since

$$f^*(v, y, \omega) = V^*(y) + \sum_{t=0}^{T-1} \delta_{\{0\}}(y \Delta s_{t+1}(\omega) + v_t),$$

this proves the claim. □

Under the mild assumptions that the price process is bounded from below and that $s_0 \in L^\infty$, we recover the following result on the existence of solutions by Rásonyi and Stettner in [17, Theorem 2.7].

Corollary 13. *Assume that the market satisfies* (NA)*, there exists $\tilde{u} < 0$ and $0 < \gamma < 1$ such that*

$$V(\lambda u) \geq \lambda^\gamma V(u) \quad \forall u \leq \tilde{u},\ \lambda \geq 1,$$

the functions $V_T(u,\omega) = V(u)$ *and*

$$V_t(u) = \text{ess inf}_{x_t \in L^0(\mathcal{F}_t)} E[V_{t+1}(u + x_t \cdot \Delta s_{t+1}) \mid \mathcal{F}_t]$$

on $\mathbb{R} \times \Omega$ *are well-defined and proper, and that* $EV_0(u) > -\infty$ *for all constants* u*. If the price process is bounded from below and* $s_0 \in L^\infty$*, then* (ALM) *has a solution for every* $u \in L^\infty$*.*

Proof. By [9, Corollary 6.1], (4) and the given growth condition for V are equivalent. Thus, by Theorem 7, Corollary 11 and Remark 3, it suffices to show that (ALM) is finite for $u = c$, where $c < 0$ is a constant.

Let $x \in \mathcal{N}$ be such that $EV(c - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}) < \infty$. The proofs of Proposition 4.2 and Proposition 4.4 in [17] show that, outside an evanescent set,[3] each

$$\tilde{V}_t(u,\omega) = E[V_{t+1}(u + x_t \cdot \Delta s_{t+1}) \mid \mathcal{F}_t](\omega)$$

and each $V_t(u,\omega)$ is convex, nondecreasing and lower semicontinuous in the u-argument and $\mathcal{F}_t$-measurable in the ω-argument. Thus these functions are normal $\mathcal{F}_t$-integrands by [21, Proposition 14.39], and it is therefore not difficult to verify that $V_t \leq \tilde{V}_t$ outside an evanescent set. Consequently,

$$V_t\left(c + \sum_{t'=0}^{t-1} x_{t'} \Delta s_{t'+1}\right) \leq E\left[V_{t+1}\left(c + \sum_{t'=0}^{t} x_{t'} \Delta s_{t'+1}\right) \middle| \mathcal{F}_t\right] \ P\text{-a.s.},$$

where the sum on the left side is defined as zero for $t = 0$. A repetition of these arguments for every t gives

$$EV_0(c) \leq EV\left(c + \sum_{t=0}^{T-1} x_t \Delta s_{t+1}\right),$$

from where we get the claim by taking the infimum over $x \in \mathcal{N}$.

□

The next example shows that some form of boundedness assumption is needed in Theorem 12. It also shows that our main theorem on the closedness of the value function cannot fully recover the main result of Rásonyi and Stettner in [17]. However, our result does not merely give the existence of solutions but also the closedness for the associated value function. The example shows that it is possible to have existence of solutions without having an equivalent martingale measure Q for which $EV^*(dQ/dP) < \infty$. To emphasize, this may happen with a finite optimal value for some extreme choices of the price process even under the asymptotic elasticity condition.

Example 2. *Let* $T = 1$*,* $\mathcal{F}_0$ *be trivial (but complete), and let* V *be unbounded from below that satisfies* (4) *and* $\text{dom}\, V = \mathbb{R}_-$*. We choose a market with a*

[3] A set in $\mathbb{R} \times \Omega$ is evanescent if its projection onto Ω is a null-set.

single asset such that $s_0 = 0$ *and* s_1^+ *and* s_1^- *are unbounded with* $EV(-\lambda s_1^+) = EV(-\lambda s_1^-) = -\infty$ *for every* $\lambda > 0$.

We have $EV(c - x_0 s_1) = +\infty$ *for every* $x \neq 0$ *and* $c \in L^\infty$, *so* $\varphi(c) = EV(c)$ *for every* $u \in L^\infty$ *and, in particular,* $\varphi(c)$ *is finite for every nonpositive* $c \in L^\infty$. *Assume now for a contradiction that there exists an equivalent martingale measure* Q *of* s *such that* $EV^*(dQ/dP) < \infty$. *Then, for the nonnegative convex normal integrand* $h^*(q,\omega) = V^*(q) + |s(\omega)|q$, *the integral functional* Eh^* *would be proper and closed on* L^1 *as would its conjugate* $(Eh^*)^*$ *on* L^∞. *By [21, Theorem 14.60],*

$$(Eh^*)^*(0) = EV(-|s|),$$

which is a contradiction, since we have chosen s *so that* $EV(-|s|) = -\infty$.

We finish by pointing out that our main results are applicable in much more general settings of optimal investment in illiquid markets as well; see [15] for such example with utilities that are bounded from above. Extensions to other illiquid models together with unbounded utilities are analyzed in [16].

4 Acknowledgements

The author is grateful to the Einstein Foundation for the financial support. The author thanks Teemu Pennanen and Miklós Rásonyi for helpful comments and discussions.

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