MEAN-VARIANCE HEDGING FOR INTEREST RATES MODELS WITH STOCHASTIC VOLATILITY

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Abstract. The mean-variance hedging approach for pricing and hedging claims in incomplete markets was originally introduced for risky assets. The aim of this paper is to apply the mean-variance hedging approach to interest rate models in presence of stochastic volatility, seen as a result of incomplete information. We set a finite number of bonds such that the volatility matrix is invertible and provide an explicit formula for the density of the variance-optimal measure which is independent by the chosen times of maturity.
Finally, we compute the mean-variance hedging strategy for a caplet and compare it with the optimal one according to the local risk minimizing approach.

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1. Introduction

The mean-variance hedging approach for pricing and hedging claims in incomplete markets was originally introduced for risky assets by several authors. Schweizer (1999) presents a general overview of the main results of the mean-variance hedging theory and a complete bibliography.
A typical example of market incompleteness is given by stochastic volatility models. For risky assets, the mean-variance hedging criterion has been analyzed in models where the volatility follows a diffusion by Laurent and Pham (1999) and where the volatility jumps by Biagini,

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Guasoni and Pratelli (2000).
The aim of this paper is to apply the mean-variance hedging approach to interest rate models in presence of stochastic volatility. Several stochastic volatility models for bonds have been proposed in literature (Longstaff and Schwartz (1992), Chiarella and Kwon (2000)). Here a stochastic volatility model is seen as a model with *incomplete information* as in the approach introduced for risky assets by Föllmer and Schweizer (1991). In a Heath-Jarrow-Morton framework, we suppose that the forward rate volatility is affected by an additional source of randomness and is measurable with respect to a filtration larger than the one available to the agent. In this setting the market is incomplete in spite of the fact that in principle an infinite number of bonds is available for trade. Since a perfect replication is not possible, we compute the variance-optimal measure's density in order to find the mean-variance optimal strategy for a given European option. We remark that we consider only self-financing portfolios composed by a finite number of bonds as in the approach of Musiela and Rutkowski (1997).

2. The Model

We introduce here our basic model. Our set of states of natures is given by the product probability space $(\Omega \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$, where $(\Omega, \mathcal{F}^W, \mathcal{F}^W_t, P^W)$ and $(E, \mathcal{E}, \mathcal{E}_t, P^E)$ are two complete filtered probability spaces. In particular, all filtrations are supposed to satisfy the so-called *"usual hypothesis"*. We assume that W_t is a standard n-dimensional Brownian motion on $\Omega = \mathcal{C}([0,T], \mathbb{R})$, P^W is the Wiener measure and $\mathcal{F}^W_t$ is the P^W-augmentation of the filtration generated by W_t.
The space E represents an additional source of randomness which affects the market. In the terminology of Föllmer and Schweizer (1991), the market is now incomplete as a result of *incomplete information*: if the evolution of η had been known the market would be complete.
We suppose that there exists on E a square integrable (eventually d-dimensional) martingale M_t endowed with the predictable representation property, i.e. for every square integrable martingale N_t there exists a predictable process H_t such that $N_t = N_0 + \int_0^t H_s dM_s$.
We analyze the mean-variance hedging criterion in the case of interest rates models. The assets to be considered on the market are zero coupon bonds with different maturities. By following the notation of Björk (1998), we denote by $p(t,T)$ the price at time t of a bond maturing at time T, where for every fixed T, the process $p(t,T)$ is an optional stochastic process such that $p(t,t) = 1$ for all t.
We assume that there exists a frictionless market for T-bonds for every

$T > 0$ and that for every fixed t, $p(t,T)$ is almost surely differentiable in the T-variable.

The *forward rate* $f(t,T)$ is defined as $f(t,T) = -\dfrac{\partial \log p(t,T)}{\partial T}$ and the *short rate* as $r_t = f(t,t)$. The *money market account* is given by the process $B_t = \exp\left(\int_0^t r_s ds\right)$.
According to the Heath-Jarrow-Morton approach (see Heath, Jarrow and Morton (1992) for further details), we describe the forward rate dynamics. In this setting, $f(t,T)$ is represented by a process on the product probability space $(\Omega \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$ such that

$$df(t,T,\eta) = \alpha(t,T,\eta)dt + \sigma(t,T,\eta)dW_t \tag{1}$$

with initial condition $f(0,T,\eta) = f^*(0,T)$. We make the following assumptions:

i) The equation (1) admits P^E-a.e. a unique strong solution with respect to the filtration $\mathcal{F}_t^W$. For example, it is sufficient that μ and σ are P^E-a.e. bounded.
ii) The information available at time t is given by the filtration $\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{E}_t$.
iii) There exists a predictable $\mathbb{R}^n$-valued process h_t such that the integral $\int h_s dW_s$ is well defined and

$$\alpha(t,T,\eta) = \sigma(t,T,\eta)\int_t^T \sigma(t,s,\eta)ds - \sigma(t,T,\eta)h_t(\eta) \tag{HJM}$$

for every $T \geq 0$. This condition is usually addressed as the *Heath-Jarrow-Morton condition* on the drift. In general, it guarantees the existence of an equivalent martingale measure for $\dfrac{p(t,T)}{B_t}$ as long as $\mathcal{E}\left(\int hdW\right)$ is a uniformly integrable martingale. In the complete market case, it is even sufficient to characterize the unique martingale measure, while in this setting of incomplete information there exists an infinite number of them.

By Proposition 15.5 of Björk (1998), we obtain the bond price dynamics:

$$\frac{dp(t,T)}{p(t,T)} = (r(t,\eta) + \frac{1}{2}\|S(t,T,\eta)\|^2 + A(t,T,\eta))dt + S(t,T,\eta)dW_t$$

where

(1) $S_i(t,T,\eta) = -\int_t^T \sigma_i(t,s,\eta)ds$
(2) $A(t,T,\eta) = -\int_t^T \alpha(t,s,\eta)ds$

Since in principle an infinite number of bonds is available for trade, one can suppose that the market is complete in spite of lack of information. This is not true since the future evolution of η cannot be predicted neither through the observation of the entire term structure. For a rigorous proof of this fact, see the doctoral dissertation of Biagini (2001).

In this setting, the market would be complete if one had access to the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_t^W \otimes \mathcal{E}$, which contains at any time all the information about past and future evolution of η. In terms of conditions on the volatility matrix, by Proposition 4.3 by Björk (1997) we obtain that this fact boils down to assume the existence of n times of maturity $T_1, \dots, T_n$, where $n = \dim W_t$, such that the matrix of elements $[A_t]_{ji} = \int_{T_0}^{T_j} \sigma_i(t,s)ds$ has rank equal to n for every $t \in [0, T_0]$ and for P^E-almost every $\eta \in E$. We assume it as standing hypothesis in the sequel. For instance, sufficient conditions implying the existence of such maturities are given by Proposition 5.5 and Theorem 5.6 of Björk, Kabanov and Runggaldier (1997).

3. The Variance-Optimal Measure for Interest Rates

In this framework, we study the problem of an agent wishing to hedge a certain European option H expiring at time T_0 by using a self-financing portfolio composed by a finite number of bonds of convenient maturities and eventually by the money market account B_t. In the sequel, for the sake of simplicity we will write $\frac{dQ}{dP}$ instead of $\frac{dQ}{dP}|_{\mathcal{F}_{T_0}}$. Since a perfect replication is not possible, we look for a solution to the minimization problem:

$$\min E\left[(H - V_{T_0})^2\right] \tag{2}$$

Usually the money market account $B_t = \exp\left(\int_0^t r(s,\eta)ds\right)$ is used as discounting factor. Now the spot rate is stochastic, so the choice of B_t as numéraire becomes unfortunate. In Sekine (1999), the impact of a stochastic interest rate is analyzed in a Markovian framework for the futures case. If the chosen discounting factor is a stochastic process, by Gouriéroux, Laurent and Pham (1998) we get that the minimization problem

$$\min E\left[(H - V_{T_0})^2\right]$$

is equivalent to

$$\min E^B\left[\left(\frac{H}{B_{T_0}} - \frac{V_{T_0}}{B_{T_0}}\right)^2\right]$$

where E^B is the expectation under the equivalent probability P^B with density

$$\frac{dP^B}{dP} = \frac{B_{T_0}^2}{E\left[B_{T_0}^2\right]}$$

In order to avoid the computation of the new bond dynamics under P^B, we can choose as numéraire the bond $p(t, T_0)$ expiring at the same time of maturity as H. We immediately have

$$\frac{dP^{T_0}}{dP} = \frac{p(T_0, T_0)^2}{E\left[p(T_0, T_0)^2\right]} = 1$$

or in other words $P^{T_0} \equiv P$.

We choose $T_1 < \cdots < T_n$ times of maturity such that such that the matrix $\int_{T_0}^{T_j} \sigma_i(t, s)ds$ is P^E-a.e. invertible for every t and set $X_t^j = \dfrac{p(t, T_j)}{p(t, T_0)}$, $j = 1, \ldots, n$. We define

$$\Theta = \left\{\theta \in L(X) : \int \theta dX \in \mathcal{S}^2\right\}$$

where $\mathcal{S}^2$ is the space of square-integrable semimartingales and $L(X)$ is the set of integrable processes with respect to X_t.

Definition 3.1. A $\mathbb{R}^{n+1}$-valued predictable process (θ_0, θ), with $\theta = (\theta_1, \ldots, \theta_n)$, is a self-financing strategy if the wealth process $V_t = \sum_{i=0}^n \theta_t^j p(t, T_j)$ satisfies $V_t = V_0 + \sum_{i=0}^n \int_0^t \theta_u^j dp(u, T_j)$. Since under the numéraire $p(t, T_0)$, the portfolio's discounted value is given by $\dfrac{V_t}{p(t, T_0)} = \dfrac{V_0}{p(0, T_0)} + \displaystyle\int_0^t \theta_u dX_u$, we assume that θ belongs to Θ.

More precisely, we are not simply interested in a self-financing portfolio whose final value has minimal quadratic distance by H, but we look for a solution to the minimization problem:

$$\min_{\substack{V_0 \in \mathbb{R} \\ \theta \in \Theta}} E\left[(H - V_0 - G_{T_0}(\theta))^2\right] \tag{3}$$

where $G_t(\theta) = \displaystyle\int_0^t \theta_s dX_s$. The space of integrals $\mathcal{G} = G_{T_0}(\Theta)$ represents the self-financing strategies with initial value $V_0 = 0$. Hence, by following Schweizer (1999), we impose on the underlying financial market the so-called *no-approximate profit condition*:

$$1 \notin \bar{\mathcal{G}} \tag{4}$$

which represents a sort of no-arbitrage condition. Here $\bar{\mathcal{G}}$ is the closure of $\mathcal{G}$ in L^2.

Problem (3) admits a unique solution (V_0, θ) for all $H \in L^2$ under the hypothesis that $G_{T_0}(\Theta)$ is closed (see Gouriéroux, Laurent and

Pham, 1998, and Rheiländer and Schweizer, 1997, for the proof). In this case, θ is called the *mean-variance optimal* strategy and V_0 the *approximation price* and they can be computed in terms of the so-called *variance-optimal measure* (Schweizer (1996), Rheiländer and Schweizer (1997)).

We denote as $\mathcal{M}^2_s(T_1, \ldots, T_n)$ and $\mathcal{M}^2_e(T_1, \ldots, T_n)$ respectively the set of *signed martingale measures* and the set of *equivalent martingale measures* for $X^j_t = \dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1, \ldots, n$.

Definition 3.2. The variance-optimal measure $\widetilde{P}^0$ is the element of $\mathcal{M}^2_s(T_1, \ldots, T_n)$ of minimal norm, where for every $Q \in \mathcal{M}^2_s(T_1, \ldots, T_n)$

$$\|\frac{dQ}{dP}\|^2 = E\left[(\frac{dQ}{dP})^2\right]$$

If $\mathcal{M}^2_s(T_1, \ldots, T_n)$ is nonempty, then $\tilde{P}$ always exists, as it is the minimizer of the norm in a convex set, and is actually an equivalent martingale measure if X_t has continuous paths (Delbaen and Schachermayer, 1996). Apparently, Definition 3.2 of $\widetilde{P}^0$ depends on the chosen maturities $T_1, \ldots, T_n$. Theorem 3.3 provides an explicit expression for the variance-optimal martingale measure's density and shows that it is actually invariant under a change of the set of maturities if the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is P^E-a.e. invertible for every t. Its proof follows by Lemma 7.3 contained in the Appendix.

Theorem 3.3. *Let H, K be two predictable processes such that the exponential martingales $\mathcal{E}\left(-\int_0^{\cdot}(h_s(\eta) + S(s,T_0,\eta))dW_s + \int_0^{\cdot} K_s dM_s\right)$ and $\mathcal{E}\left(\int_0^{\cdot}(-h_s(\eta) - S(s,T_0,\eta) + H_s)d\widehat{W_s}\right)$ are square-integrable. Then*

$$\frac{d\widetilde{P}^0}{dP} = \mathcal{E}\left(-\int_0^{\cdot}(h_s(\eta) + S(s,T_0,\eta))dW_s + \int_0^{\cdot} K_s dM_s\right)_{T_0} \tag{5}$$

or equivalently

$$\frac{d\widetilde{P}^0}{dP} = \frac{\mathcal{E}\left(-\int_0^{\cdot}\beta_s dX_s\right)_{T_0}}{E\left[\mathcal{E}\left(-\int_0^{\cdot}\beta_s dX_s\right)_{T_0}\right]} \tag{6}$$

where H, K are solutions of the equation (13) of Lemma 7.3 and $\beta^j_s = \dfrac{p(s,T_0)}{p(s,T_j)}\displaystyle\sum_i (h^i_s(\eta) + S^i(s,T_0,\eta)) - H^i_s)[A_s^{-1}]_{ij}$.

In particular, if $\sigma(t,T,\eta,\omega) = \sigma(t,T,\eta)$, by Biagini, Guasoni and Pratelli (2000) we obtain that the density of $\widetilde{P}$ has the form

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_s dW_s\right)_{T_0} \frac{\exp\left(-\int_0^{T_0} \|\lambda_s\|^2 ds\right)}{E\left[\exp\left(-\int_0^{T_0} \|\lambda_s\|^2 ds\right)\right]} \tag{7}$$

where $\lambda_t = h_t(\eta) + S(t, T_0, \eta)$.

4. Examples

Here we illustrate some examples where the stochastic volatility is a consequence of incomplete information and show how to construct the additional probability space $(E, \mathcal{E}, \mathcal{E}, P^E)$ and the martingale M_t on E with the representation property.
The Heath-Jarrow-Morton condition on the drift allows us to model only the forward rate volatility $\sigma(t, T, \eta)$ and $h_t(\eta)$. Without loss of generality, we suppose that $h_t(\eta)$ is affected by the same behavior as $\sigma(t, T)$ and we model only the volatility $\sigma(t, T)$.

Example 4.1. First we consider the case when $\dim W_t = 1$ and

$$\sigma(t, T) = \sigma_0 I_{\{t<\eta, t\leq T\}} + \sigma_1 I_{\{t\geq\eta, t\leq T\}}$$

where $\sigma_0, \sigma_1 \in \mathbb{R}^+$ and η is a totally inaccessible stopping time. Here we set $E = \mathbb{R}^+$, $\mathcal{E}_t = \mathcal{B}([0, t]) \vee (t, +\infty]$ and a fundamental martingale is given by $M_t = I_{\{t\geq\eta\}} - a_t$, where a_t is the compensator of the process $I_{\{t\geq\eta\}}$ associated to η.
More generically η can be assumed to be a Markov process η_t in continuous time with a finite set of states I. This example models the situation when the volatility has multiple jumps occurring at independent random times.

Example 4.2. If in Example 4.1 the volatility assumes values after the jump according to a general probability distribution, there does not exist a finite set of martingales with the predictable representation property. By following Jacod and Shiryaev (1987), we can substitute M_t with the compensated integer-valued random measure $\mu - \nu$ associated to η_t, which has the predictable representation property with respect to the smallest filtration under which μ is optional.

Example 4.3. Finally η_t can be given by a diffusion process

$$\begin{aligned} df(t,T) &= \alpha(t,T,\eta_t)dt + \sigma(t,T,\eta_t)dW_t^1 \\ d\eta_t &= F(t,T,\eta_t)dt + G(t,T,\eta_t)dW_t^2 \end{aligned}$$

where W_t^1 can be correlated with W_t^2. This example has been studied in the case of risky assets by using dynamic programming techniques in Laurent and Pham (1999). Clearly, here $M_t = W_t^2$.

5. Mean-Variance Hedging for a Call Option

As in Biagini and Guasoni (1999), we suppose now that $\sigma(t,T,\omega,\eta) = \sigma(t,T,\eta)$. We remark that in this particular case the variance-optimal density is given by (7). We compute now the mean-variance optimal strategy for a call option expiring at time T_0 on a T_1-bond ($T_0 < T_1$) by exploiting the explicit characterization for the variance-optimal measure's density provided by Theorem 3.3. Let $T_1 < T_2 < \cdots < T_n$ be maturities such that the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is P^E-a.e. invertible for every t. If there exists at least an equivalent martingale measure for $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1,\ldots,n$, and the space of integrands $G_{T_0}(\Theta)$ is closed, the variance-optimal strategy for the call option $H = (P(T_0,T_1) - K)^+$ is given by the following

Proposition 5.1. *If $p(T_0,T_1)$ is square-integrable with respect to $\tilde{P}$, the components θ^j of the variance-optimal strategy are given in the following feedback form:*

(1) *for $j = 1$*

$$\theta_t^1 = \xi_t^1 - \beta_t^1 (\xi_t^1 \frac{p(t,T_1)}{p(t,T_0)} - K\xi_t^0 - \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \right] - \int_0^t \theta_s dX_s)$$

(2) *for $j > 1$*

$$\theta_t^j = -\beta_t^j (\xi_t^1 \frac{p(t,T_1)}{p(t,T_0)} - K\xi_t^0 - \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \right] - \int_0^t \theta_s dX_s)$$

where $\xi_t^0 = -K\tilde{E}^0 \left[1_A | \mathcal{F}_{t-} \right]$, $\tilde{\xi}_t^1 = \tilde{E}^1 \left[1_A | \mathcal{F}_{t-} \right]$ with $A = \{p(T_0,T_1) \geq K\}$ and $\beta_t^j = \dfrac{p(t,T_0)}{p(t,T_j)} \sum_i (h_t^i(\eta) + S^i(t,T_0,\eta))[A_t^{-1}]_{ij}$.

Proof. We need to compute all terms in the implicit characterization of the mean-variance optimal strategy given in Theorem 6 by Rheiländer

and Schweizer (1997).
By Schweizer (1996), it follows that $c = \tilde{E}^0\left[(p(T_0,T_1)-K)^+\right]$.
By Theorem 3.3 we obtain that

$$\frac{\tilde{\zeta}_t^j}{\tilde{Z}_t} = \beta_t^j = \frac{p(t,T_0)}{p(t,T_j)}\sum_i (h_t^i + S^i(t,T_0,\eta))[A_t^{-1}]_{ij}$$

In order to compute ξ_t^j, note that with respect to the "enlarged" filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{E}_{T_0}$ that contains all information about η, H can be perfectly replicated by the self-financing portfolio $\tilde{E}^0\left[H\left|\tilde{\mathcal{F}}_t\right.\right]$. If we denote $\tilde{\xi}_t^j$ the portfolio component with respect to $p(t,T_j)$, by the standard theory of complete markets we obtain $\tilde{\xi}_t^0 = -K\tilde{E}^0\left[1_A\left|\tilde{\mathcal{F}}_t\right.\right]$, $\tilde{\xi}_t^1 = \tilde{E}^1\left[1_A\left|\tilde{\mathcal{F}}_t\right.\right]$ and $\tilde{\xi}_t^j = 0$ for every $j \neq 0,1$. ξ_t^i is given by the predictable projection of $\tilde{\xi}_t^i$ with respect to $\mathcal{F}_t$, i.e. $\xi_t^0 = -K\tilde{E}^0\left[1_A\right|\mathcal{F}_{t-}]$, $\xi_t^1 = \tilde{E}^1\left[1_A\right|\mathcal{F}_{t-}]$ and $\xi_t^i = 0$ for every $i \neq 0,1$. For further details, see the doctoral dissertation of Biagini (2001).
Moreover, by applying these results and the change of numéraire technique introduced in Geman, El Karoui and Rochet (1995) to $\tilde{V}_t = \tilde{E}^0\left[(p(T_0,T_1)-K)^+\right|\mathcal{F}_t]$, we obtain:

$$\tilde{V}_{t-} = p(t,T_1)\tilde{E}^1\left[1_A\right|\mathcal{F}_{t-}] - Kp(t,T_0)\tilde{E}^0\left[1_A\right|\mathcal{F}_{t-}]$$

where $A = \{p(T_0,T_1) \geq K\}$. □

We remark that the mean-variance optimal strategy depends on a number of bonds equal to $(\dim W_t + 1)$.
We apply these results in order to price and hedge the caplet $H = \delta(\frac{1-p(T_0,T_1)}{\delta p(T_0,T_1)} - R)^+$ in this framework of incomplete information by using the mean-variance hedging approach. We refer to Björk (1997) for all definitions and properties concerning the caplets.
Since the caplet is settled in arrears, we consider H as a T_1-option and we choose $p(t,T_1)$ as discounting factor. The approximation price of H is equal to $\tilde{E}^1[H]$, where the expectation is done under the variance-optimal measure with $p(t,T_1)$ as numéraire. The caplet can be written as

$$H = \frac{R^*}{p(T_0,T_1)}(\frac{1}{R^*} - p(T_0,T_1))^+$$

where $R^* = 1 + \delta R$. The approximation price is given by

$$\tilde{E}^1\left[H\right] = R^*\tilde{E}^1\left[\frac{1}{p(T_0,T_1)}(\frac{1}{R^*} - p(T_0,T_1))^+\right] \tag{8}$$

Since

$$\frac{d\widetilde{P}^1}{d\widetilde{P}^0}\Big|_{\mathcal{F}_{T_0}} = \frac{p(T_0,T_1)}{p(T_0,T_0)} \cdot \frac{p(0,T_0)}{p(0,T_1)} \tag{9}$$

we can exploit in (8) the change of numéraire technique obtaining

$$\tilde{E}^1\left[H\right] = R^*\tilde{E}^0\left[(\frac{1}{R^*} - p(T_0,T_1))^+\right]$$

H has the same approximation price of R^* put options $K = (\frac{1}{R^*} - p(T_0,T_1))^+$ on $p(t,T_1)$ expiring at time T_0.

Remark 5.2. Since H is actually $\mathcal{F}_{T_0}$-measurable, a natural question is whether the mean-variance optimal strategy up to time T_0 for T_1-option H coincides with (R^* times) the one for the T_0-put option K as in the complete market case (see Björk, 1997). The answer is negative as expected since $\Theta_{T_0} \subseteq \Theta_{T_1}$.

We fix $(n+1)$ bonds $p(t,T_1), p(t,T_2), \dots, p(t,T_{n+1})$ such that the matrix $[B_t]_{ji} = \int_{T_1}^{T_j} \sigma_i(t,s)ds$ is invertible for every t P^E-almost everywhere and put $Y_t^j = \frac{p(t,T_j)}{p(t,T_1)}$, $j = 2,\dots,n+1$. Note that in order to compute the two mean-variance hedging strategies, we need to use the same assets for both. Consequently, we cannot choose $p(t,T_0)$ since it is not defined after $t = T_0$. All computations are done under $p(t,T_1)$ as numéraire. We set $\tilde{V}_t = p(t,T_1)\tilde{E}^1\left[H|\,\mathcal{F}_t\right]$. Recall that $\tilde{V}_t = R^*p(t,T_1)\tilde{E}^1\left[\frac{K}{p(T_0,T_1)}\middle|\,\mathcal{F}_t\right]$.

By Proposition 5.1, we obtain that

(1) for $j > 2$, the optimal components for H as T_1-option are given by

$$\theta_t^{j,1} = -\beta_t^j(\tilde{V}_{t-} - \tilde{V}_0 - \int_0^t \theta_s^1 dY_s)$$

where $\beta_t^j = \frac{p(t,T_1)}{p(t,T_j)}\sum_i (h_t^i + S(t,T_0,\eta))[B_t^{-1}]_{ij}$.

(2) for $j > 2$, the optimal components for the T_0-option R^*K are given by

$$\theta_t^{j,0} = -(\beta_t^j + \gamma_t^1)(\tilde{V}_{t-} - \tilde{V}_0 - \int_0^t \theta_s^0 dY_s)$$

where γ_t^1 is the solution of the following equation

$$\frac{dP^{T^1}}{dP} = \frac{p(T_0,T_1)^2}{E\left[p(T_0,T_1)^2\right]} = \mathcal{E}\left(\int_0^\cdot \gamma_s^1 dW_s + \int_0^\cdot \gamma_s^2 dM_s\right)_{T_0}$$

since the use of $p(t,T_1)$ as discounting factor for a T_0-option compels us to work under the probability P^{T_1}.

We can easily conclude that two strategies do not coincide up to time $t = T_0$ unless $\gamma_t^1 = 0$, which is in general not the case.

In order to compute an approximation strategy for the caplet, we can proceed like in the complete market case (see Björk, 1997). We find the variance-optimal portfolio for the T_0-put option $K = (\frac{1}{R^*} - p(T_0,T_1))^+$ and invest the final value V_{T_0} in $p(t,T_1)$ from time $t = T_0$ to T_1. As shown in Remark 5.2, this strategy is not the optimal one for H. Nevertheless, this method results of some interest since the strategy can be computed in terms of the "natural" assets $p(t,T_0)$ and $p(t,T_1)$ and the approximation price $\tilde{E}^0[K]$ for K coincides with the approximation price $\tilde{E}^1[H]$ for H.

6. A comparison with the local risk minimizing approach

An alternative approach in order to price and hedge a contingent claim in the incomplete market case is represented by the local risk minimization one (for a complete treatment of the subject, see Schweizer, 1999). The main difference with respect to mean-variance hedging is that a local risk minimizing strategy perfectly replicates the value of a given option, but it is not self-financing. More precisely, suppose we want to hedge a T_0-option H by using a portfolio based on a finite number of bonds $p(t,T_0), p(t,T_1), \ldots, p(t,T_n)$ such that the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is P^E-a.e. invertible for every t. As in the previous sections, we set $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1,\ldots,n$. By exploiting the approach of Biagini and Pratelli (1999), we have the following

Definition 6.1. An L^2-*strategy* is a pair (θ,θ^0) such that $\theta \in \Theta = \left\{\theta \in L(X) : \int \theta dX \in \mathcal{S}^2\right\}$ and θ^0 is a real predictable process such that the *value process* left limit $V_{t-} = \theta_t \cdot X_t + \theta_t^0$ is square integrable for $0 \le t \le T_0$.

The (cumulative) *cost process* is defined by $C_t = V_t - \int_0^t \theta_s dX_s$, $0 \leq t \leq T_0$.

By Definition 6.1, we get that the portfolio's jumps coincide with the jumps in the cost process. In a self-financing portfolio, the cost is constant.

Definition 6.2. Let $H \in L^2(\mathcal{F}_{T_0}, P)$ be a contingent claim. An L^2-strategy (θ, θ^0) with $V_{T_0} = H \ P - a.s.$ is called *pseudo-locally risk-minimizing* or *pseudo-optimal* for H if the cost process C_t is a P-martingale and is strongly orthogonal to the martingale part of X.

We remark that the optimal strategy is invariant under a change of numéraire (for more details, see Biagini and Pratelli, 1999).
By Definition 6.2, it immediately follows that a contingent claim $H \in L^2(\mathcal{F}_{T_0}, P)$ admits a pseudo-optimal strategy if and only if H can be written as

$$H = H_0 + \int_0^{T_0} \xi_u dX_u + L_{T_0} \tag{10}$$

where $H_0 \in L^2(\mathcal{F}_{T_0}, P)$, $\xi \in \Theta$ and L is a square integrable martingale strongly P-orthogonal to the martingale part of X. Equation (10) is usually addressed in literature as the *Föllmer-Schweizer decomposition* of H. It is connected to a suitably chosen martingale measure, the so-called *minimal martingale measure.*

Definition 6.3. $\widehat{P}^0 \in \mathcal{M}_e^2(T_1, \ldots, T_n)$ is the *minimal measure* (with respect to $p(t, T_0)$ as numéraire) if any locally square integrable local martingale which is orthogonal to the martingale part of X under P remains a local martingale under $\widehat{P}^0$.

By Definition 6.3 follows immediately that the pseudo-optimal portfolio $\widehat{V}(\phi)$ is a local $\widehat{P}^0$-martingale and we get

$$\widehat{V}_t(\phi) = p(t, T_0)\widehat{E}^0 \left[H | \mathcal{F}_t\right]$$

The optimal portfolio is a true martingale if $\widehat{Z}_t = \widehat{E}^0 \left[\left. \frac{d\widehat{P}^0}{dP} \right| \mathcal{F}_t \right]$ is itself a square-integrable martingale. By exploiting the results of Theorem 3.3, we obtain that

$$\frac{d\widehat{P}^0}{dP} = \mathcal{E}\left(- \int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s \right)_{T_0}$$

define the minimal measure's density as long as the Doleans Exponential $\mathcal{E}\left(-\int_0^{\cdot}(h_s(\eta)+S(s,T_0,\eta))dW_s\right)$ is a uniformly integrable martingale.
The pseudo-optimal strategy for a call option in presence of incomplete information has been computed in Biagini and Pratelli (1999) for the risky assets case. Their results can be easily extended to the interest rate case. By Theorem 5.1 of Biagini and Pratelli (1999), we obtain that the pseudo-optimal portfolio is given by

$$\widehat{V}_t(\phi) = p(t,T_0)\widehat{E}^0\left[H|\,\mathcal{F}_t\right] = p(t,T_1)\widehat{E}^1\left[I_A|\,\mathcal{F}_t\right] - Kp(t,T_0)\widehat{E}^0\left[I_A|\,\mathcal{F}_t\right]$$

and the optimal strategy components are $\theta_t^0 = -K\widehat{E}^0\left[1_A|\,\mathcal{F}_{t-}\right]$, $\theta_t^1 = \widehat{E}^1\left[1_A|\,\mathcal{F}_{t-}\right]$ and $\theta_t^j = 0$ for all $j = 2,\dots,n$. Note that in the local risk minimization case, the pseudo-optimal strategy depends only on two assets in spite of the dimension of the driving Brownian motion. On the contrary, the mean-variance optimal strategy is based on $(n+1)$ bonds, where $n = \dim W_t$.
We apply these results in order to compute the local risk minimizing strategy for a caplet. In the same notation of the previous section, the pseudo-optimal portfolio for the caplet $H = \dfrac{R^*}{p(T_0,T_1)}(\dfrac{1}{R^*} - p(T_0,T_1))^+$ is given by $\widehat{V}_t = p(t,T_1)\widehat{E}^1\left[H|\,\mathcal{F}_t\right]$ which for $t \leq T_0$ coincides with the optimal portfolio for the T_0-put option $K = (\dfrac{1}{R^*} - p(T_0,T_1))^+$ since by Theorem 3.2 of Biagini and Pratelli (1999) we have

$$\frac{d\widehat{P}^1}{d\widehat{P}^0} = p(T_0,T_1)\frac{p(0,T_0)}{p(0,T_1)}$$

For $t > T_0$, $\widehat{V}_t = \widehat{E}\left[H|\,\mathcal{F}_t\right] = H$ since H is $\mathcal{F}_{T_0}$-measurable. In other words, the pseudo-optimal portfolio for H is constant after $t = T_0$. Consequently, in the local risk-minimization case the strategies for the T_1-option H and for the T_0-option K coincide up to time T_0 and we can behave exactly as in the complete market case. The key is that in this approach we perfectly replicate the option value in spite of approximating it as in the mean-variance hedging criterion.

7. Appendix

Here we simply sketch how to find an explicit characterization of $\widetilde{P}^0$ in order to solve the mean-variance hedging problem in the interest-rate case. In the doctoral dissertation of Biagini (2001), all computations are done to full extent.
In order to obtain an explicit formula for the variance-optimal measure,

first we characterize the martingale measures for $(\frac{p(t,T)}{p(t,T_0)})_{t\in[0,T_0]}$ for every $T>0$.

Lemma 7.1. *Let Z_t be a local martingale with $Z_0=1$. The following conditions are equivalent:*

(1) $Z_t\frac{p(t,T)}{p(t,T_0)}$ *is a local martingale for every $T>0$*
(2) $Z_t = \mathcal{E}\left(-\int_0^{\cdot}(h_s+S(s,T_0,\eta))dW_s\right)_t(1+\int_0^t k_s dM_s)$ *for some predictable process k_s such that the integral $\int_0^t k_s dM_s$ is a local martingale.*

Proof. (W_t,M_t) has the representation property on $(\Omega\times E,\mathcal{F}\otimes\mathcal{E},P^W\otimes P^E)$, hence there exist predictable processes λ_t and k_t such that

$$Z_t = 1+\int_0^t \lambda_s dW_s + \int_0^t k_s dM_s$$

(see for example Biagini, Guasoni and Pratelli, 2000). By applying Itô's formula, we get that the process $Z_t\frac{p(t,T)}{p(t,T_0)}$ is a local martingale if and only if the process λ_t solves the following equation for every $T>0$:

$$Z_{t-}\sum_i(h_t^i(\eta)+S_i(t,T_0,\eta))\int_{T_0}^T\sigma_i(t,s,\eta)ds+\lambda_t\int_{T_0}^T\sigma_i(t,s,\eta)ds=0 \tag{11}$$

Since we assume that there exist $T_1,\ldots,T_n$ such that the matrix $\int_{T_0}^{T_j}\sigma_i(t,s)ds$ is P^E-a.e. invertible for every t, it follows immediately that

$$\lambda_t^i = -(h_t^i(\eta)+S_i(t,T_0,\eta))$$

for $i=1,\ldots,n$.

□

By equation (11) follows immediately that the set $\mathcal{M}_s^2(T_1,\ldots,T_n)$ of martingale measures for $\frac{p(t,T_j)}{p(t,T_0)}$, $j=1,\ldots,n$, coincides with the set $\mathcal{M}_s^2(T)$ of martingale measures for $\frac{p(t,T)}{p(t,T_0)}$, $T\geq 0$. We synthesize it in the following

Proposition 7.2. (1) *If $Q \in \mathcal{M}_s^2(T_1, \ldots, T_n)$, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s\right)_{T_0} (1 + \int_0^{T_0} k_s dM_s)$$

for some predictable process k_t such that the above expression is square integrable.

(2) *If $Q \in \mathcal{M}_e^2(T_1, \ldots, T_n)$, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s\right)_{T_0} \mathcal{E}\left(\int_0^{\cdot} k_s dM_s\right)_{T_0}$$

for some predictable process k_t such that $k_t \cdot \Delta M_t > -1$ and $\mathcal{E}\left(-\int_0^{\cdot}(h_s(\eta) + S(s, T_0, \eta))dW_s + \int_0^{\cdot} k_s dM_s\right)_t$ is a square-integrable martingale.

We define the two process $\widehat{W}_t$ and W_t^* as follows:

$$\widehat{W}_t = W_t + \int_0^t (h_s(\eta) + S(s, T_0, \eta))ds$$
$$W_t^* = W_t + 2\int_0^t (h_s(\eta) + S(s, T_0, \eta))ds$$

Lemma 7.3 is quite technical, but together with Proposition 7.2 let us write an explicit expression for the density of the variance-optimal measure. Its proof is formally analogous to the one of Lemma 1.15 of Biagini, Guasoni and Pratelli (2000).

Lemma 7.3. *Let H, K be two predictable stochastic processes whose stochastic integrals $\int_0^t H_s dW_s^*$ and $\int_0^t K_s dM_s$ are defined. The following conditions are equivalent:*

$$\exp\left(\int_0^T \|(h_s(\eta) + S(s, T_0, \eta))\|^2 ds\right) = c\,\frac{\mathcal{E}\left(\int_0^{\cdot} H_s dW_s^*\right)_T}{\mathcal{E}\left(\int_0^{\cdot} K_s dM_s\right)_T} \tag{12}$$

$$\mathcal{E}\left(-\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s + \int_0^{\cdot} K_s dM_s\right)_T = \\ = c\,\mathcal{E}\left(\int_0^{\cdot} (-h_s(\eta) - S(s, T_0, \eta) + H_s)d\widehat{W}_s\right)_T \tag{13}$$

where c is the same constant in both equations.

We obtain the proof of Theorem 3.3 in the following way. Equation (13) characterizes completely the variance-optimal measure $\widetilde{P}^0$ since by Schweizer (1996) it is the unique martingale measure which can be written in the form $\mathcal{E}\left(\int \beta dX\right)$, where $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$. By using the equivalence stated in Lemma 7.3 we solve instead equation (12): since a solution (H,K) always exists because of the representation property of (W_t, M_t) on $(\Omega \times E, \mathcal{F} \otimes \mathcal{E}, P^W \otimes P^E)$, we obtain equation (5) for the variance-optimal measure's density.

References

[1] Biagini F.(2001): " Quadratic Hedging Approaches For Interest Rate Models With Stochastic Volatility", *Tesi di Perfezionamento*, Scuola Normale Superiore di Pisa.

[2] Biagini F. and Guasoni P. (2001): " Mean-Variance Hedging with Random Volatility Jumps", to appear on *Stochastic Analysis and Applications* **19**, n.2.

[3] Biagini F., Guasoni P., Pratelli M. (2000): " Mean-variance hedging for Stochastic Volatility Models ", *Mathematical Finance* **10**, n.2, 109-123

[4] Biagini F., Pratelli M. (1999): " Local Risk Minimization and Numéraire ", *Journal of Applied Probability* **36**, n.4, 1-14.

[5] Björk T.,(1997): "Interest Rate Theory". In: *Proceedings CIME Conference*, Bressanone 1996, Springer, Berlin Heildelberg New York

[6] Björk T.,(1998): "Arbitrage Theory in Continuous Time", Oxford University Press

[7] Björk T., Kabanov Y., Runggaldier W.(1997): "Bond market structure in the presence of marked point processes", *Mathematical Finance* **7**, n.2, 211-239

[8] Chiarella C., Kwon O.K.. (2000): " A class of Heath-Jarrow-Morton term structure models with stochastic volatility", Working Paper, Quantitative Finance Research Group, University of Technology Sydney, **39**

[9] Föllmer H. and Schweizer M. (1991): "Hedging of contingent claims under incomplete information". In: R.J. Elliot and M.H.A. Davis, editors, *Applied Stochastic Analysis* , Gordon and Breach, 389-414.

[10] Geman H., El Karoui N. and Rochet J.C. (1995): "Changes of numéraire, changes of probability measures and option pricing", *Journal of Applied Probability* **32**, 443-458

[11] Gouriéroux L., Laurent J.P. and Pham H. (1998): "Mean-variance hedging and numéraire" *Mathematical Finance* **8**, 179-200.

[12] Heath D., Jarrow R. and Morton A., (1992): " Bond Pricing and the Term Structure of Interest Rate", *Econometrica* **60**, 1, 77-106.

[13] Jacod J., Shiryaev N.A.,(1987): "Limit Theorems for Stochastic Processes", Springer Verlag, Berlin Heidelberg New York

[14] Laurent J.P. and Pham H. (1999): "Dynamic programming and mean-variance hedging", *Finance and Stochastics* **3(1)**, 83-110

[15] Longstaff F.A., Schwartz E.S,(1992): "A two-factor interest rate model and contingent claim valuation", *The Journal of Fixed Income*, **3(1)**, 16-23

[16] Musiela M., Rutkowski M.(1997): "Martingale Methods in Financial Modelling", Springer, Berlin Heidelberg New York

[17] Rheinländer T. and Schweizer M. (1997): "On l^2-projections on a space of stochastic integrals" *Annals of Probability* **25 (4)**, 1810-1831

[18] Sekine J.(1999): "Mean-Variance Hedging in Continuous Time with Stochastic Interest Rates",*Stochastics and Stochastic Reports* **67**, 1-16

[19] Schweizer M. (1996): "Approximation pricing and the variance-optimal martingale measure" *Annals of Probability* **64**, 206-236.

[20] Schweizer M. (1999): "A Guided Tour through Quadratic Hedging Approaches" Preprint.