Malliavin Calculus Applied to Optimal Control of Stochastic Partial Differential Equations with Jumps

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Abstract

In this paper we employ Malliavin calculus to derive a general stochastic maximum principle for stochastic partial differential equations with jumps under partial information. We apply this result to solve an optimal harvesting problem in the presence of partial information. Another application pertains to portfolio optimization under partial observation.

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1 Introduction

In this paper we aim at using Malliavin calculus to prove a general stochastic maximum principle for stochastic partial differential equations (SPDE's) with jumps under partial information. More precisely, the controlled process is given by a quasilinear stochastic heat equation driven by a Wiener process and a Poisson random measure. Further the control processes are assumed to be adapted to a subfiltration of the filtration generated by the driving noise of the controlled process. Our paper is inspired by ideas developed in Meyer-Brandis, Øksendal & Zhou [14], where the authors establish a general stochastic maximum principle for SDE's based on Malliavin calculus. The results obtained in this paper can be considered a generalization of [14] to the setting of SPDE's.

There is already a vast literature on the stochastic maximum principle. The reader is e.g. referred to [2, 3, 1, 9, 20, 17, 21] and the references therein. Let us mention that the authors in [2, 20], resort to stochastic maximum principles to study partially observed optimal control problems for diffusions, that is the controls under consideration are based on noisy observations described by the state process. Our paper covers the partial observation case in

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[2, 3, 20], since we deal with controls being adapted to a general subfiltration of the underlying reference filtration. Further, our Malliavin calculus approach to stochastic control of SPDE's allows for optimization of very general performance functionals. Thus our method is useful to examine control problems of non-Markovian type, which cannot be solved by stochastic dynamic programming. Another important advantage of our technique is that we may relax the assumptions on our Hamiltonian, considerably. For example, we do not need to impose concavity on the Hamiltonian. See e.g. [17, 1]. We remark that the authors in [1] prove a sufficient and necessary maximum principle for partial information control of jump diffusions. However, their method relies on an adjoint equation which often turns out to be unsolvable.

We shall give an outline of our paper: In Section 2 we introduce a framework for our partial information control problem. Then in Section 3 we prove a general (sufficient and necessary) maximum principle for SPDE's by invoking Malliavin calculus. See Theorem 4. In Section 4 we use the results of the previous section to solve a partial information optimal harvesting problem (Theorem 6). Further we inquire into a portfolio optimization problem under partial observation. The latter problem boils down to a partial observation problem of jump diffusions, which cannot be captured by the framework of [14].

2 Framework

In the following, let $\{B_s\}_{0\leq s\leq T}$ be a Brownian motion and $\widetilde{N}(dz,ds)=N(dz,ds)-ds\nu(dz)$ a compensated Poisson random measure associated with a Lévy process with Lévy measure ν on the (complete) filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},P)$. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0} z^2\nu(dz)<\infty,$$

where $\mathbb{R}_0:=\mathbb{R}-\{0\}$.

Consider the controlled stochastic reaction-diffusion equation of the form

$$\begin{aligned}
d\Gamma(t,x)=&\Big[L\Gamma(t,x)+b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\Big]dt\\
&+\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)dB(t)\\
&+\int_{\mathbb{R}}\theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega)\widetilde{N}(dz,dt), \qquad (2.1)\\
&(t,x)\in[0,T]\times G
\end{aligned}$$

with boundary condition

$$\begin{aligned}
\Gamma(0,x)=&\xi(x),\ x\in\overline{G}\,,\\
\Gamma(t,x)=&\eta(t,x),\ (t,x)\in(0,T)\times\partial G.
\end{aligned}$$

Here L is a partial differential operator of order m and ∇_x the gradient acting on the space variable $x \in \mathbb{R}^n$ and $G \subset \mathbb{R}^n$ is an open set. Further

$$\begin{aligned}
b(t,x,\gamma,\gamma',u,\omega) &: [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R} \\
\sigma(t,x,\gamma,\gamma',u,\omega) &: [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R} \\
\theta(t,x,\gamma,\gamma',u,z,\omega) &: [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \mathbb{R}_0 \times \Omega \longrightarrow \mathbb{R} \\
\xi(x) &: \overline{G} \longrightarrow \mathbb{R} \\
\eta(t,x) &: (0,T) \times \partial G \longrightarrow \mathbb{R}
\end{aligned}$$

are Borel measurable functions, where $U \subset \mathbb{R}$ is a closed convex set. The process

$$u : [0,T] \times G \times \Omega \longrightarrow U$$

is called an *admissible control* if (2.1) has a unique (strong) solution $\Gamma = \Gamma^{(u)}$ such that $u(t,x)$ is adapted with respect to a subfiltration

$$\mathcal{E}_t \subset \mathcal{F}_t,\ 0 \leq t \leq T\,, \tag{2.2}$$

and such that

$$\mathbb{E}\left[\int_0^T \int_G |f(t,x,\Gamma(t,x),u(t,x),\omega)|\,dx\,dt + \int_G |g(x,\Gamma(T,x),\omega)|\,dx\right] < \infty$$

for some given C^1 functions that define the performance functional (see (2.3) below)

$$\begin{aligned}
f &: [0,T] \times G \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}\,, \\
g &: G \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}.
\end{aligned}$$

A sufficient set of conditions, which ensures the existence of a unique strong solution of (2.1), is e.g. given by the requirement that the coefficients b, σ, θ satisfy a certain linear growth and Lipschitz condition and that the operator L is bounded and coercive with respect to some Gelfand triple. For more general information on the theory of SPDE's the reader may consult e.g. [6], [11].

Note that one possible subfiltration $\mathcal{E}_t$ in (2.2) is the δ-delayed information given by

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+};\ \ t \geq 0$$

where $\delta \geq 0$ is a given constant delay.

The σ-algebra $\mathcal{E}_t$ can be interpreted as the entirety of information at time t the controller has access to. We shall denote by $\mathcal{A} = \mathcal{A}_{\mathcal{E}}$ the class of all such admissible controls.

For admissible controls $u \in \mathcal{A}$ define the *performance functional*

$$J(u) = \mathbb{E}\left[\int_0^T \int_G f(t,x,\Gamma(t,x),u(t,x),\omega)\,dx\,dt + \int_G g(x,\Gamma(T,x),\omega)\,dx\right]. \tag{2.3}$$

The optimal control problem is to find the maximum and the maximizer of the performance, i.e. determine the *value* $J^* \in \mathbb{R}$ and the *optimal control* $u^* \in \mathcal{A}$ such that

$$J^* = \sup_{u \in \mathcal{A}} J(u) = J(u^*) \tag{2.4}$$

3 A Generalized Maximum Principle for Stochastic Partial Differential Equations with Jumps

In this Section we want to derive a general stochastic maximum principle by means of Malliavin calculus. To this end, let us briefly review some basic concepts of this theory. As for definitions and further information on Malliavin calculus see e.g. [16] or [7].

3.1 Some Elementary Concepts of Malliavin Calculus for Lévy Processes

Suppose that B_t is a Brownian motion on the filtered probability space

$$(\Omega^{(1)}, \mathcal{F}^{(1)}, \{\mathcal{F}_t^{(1)}\}_{0\leq t\leq T}, P^{(1)}),$$

where $\{\mathcal{F}_t^{(1)}\}_{0\leq t\leq T}$ is the $P^{(1)}-$augmented filtration generated by B_t with $\mathcal{F}^{(1)} = \mathcal{F}_T^{(1)}$. Analogously, assume a stochastic basis

$$(\Omega^{(2)}, \mathcal{F}^{(2)}, \{\mathcal{F}_t^{(2)}\}_{0\leq t\leq T}, P^{(2)})$$

associated with the compensated Poisson random measure $\widetilde{N}(dt, dz)$.

Let us recall the chaos representation property of square integrable functionals of B_t and $\widetilde{N}(dt, dz)$:

(i) If $F \in L^2(\mathcal{F}^{(1)}, P^{(1)})$ then

$$F = \sum_{n\geq 0} I_n^{(1)}(f_n) \tag{3.1}$$

for a unique sequence of symmetric $f_n \in L^2(\lambda^n)$, where λ is the Lebesgue measure and

$$I_n^{(1)}(f_n) := n! \int_0^T \int_0^{t_n} \cdots \left(\int_0^{t_2} f_n(t_1, \cdots, t_n) dB(t_1) \right) dB(t_2) \cdots dB(t_n), \quad n \in \mathbb{N}$$

the n-fold iterated stochastic integral with respect B_t. Here $I_n^{(1)}(f_0) := f_0$ for constants f_0.

(ii) Similarly, if $G \in L^2(\mathcal{F}^{(2)}, P^{(2)})$, then

$$G = \sum_{n\geq 0} I_n^{(2)}(g_n), \tag{3.2}$$

for a unique sequence of kernels g_n in $L^2((\lambda\times\nu)^n)$, which are symmetric w.r.t. $(t_1, z_1), \cdots, (t_n, z_n)$. Here $I_n^{(2)}(g_n)$ is given by

$$I_n^{(2)}(g_n) := n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \cdots \left(\int_0^{t_2} \int_{\mathbb{R}_0} g_n(t_1, z_1, \cdots, t_n, z_n) \right) \widetilde{N}(dt_1, dz_1) \cdots \widetilde{N}(dt_n, dz_n),$$
$$n \in \mathbb{N}.$$

It follows from the Itô isometry that

$$\|F\|^2_{L^2(P^{(1)})} = \sum_{n\geq 0} n!\, \|f_n\|^2_{L^2(\lambda^n)}$$

and

$$\|G\|^2_{L^2(P^{(2)})} = \sum_{n\geq 0} n!\, \|g_n\|^2_{L^2((\lambda\times\nu)^n)}\,.$$

Definition 1 (Malliavin derivatives D_t and $D_{t,z}$)

(i) *Denote by $\mathbb{D}^{(1)}_{1,2}$ the stochastic Sobolev space of all $F \in L^2(\mathcal{F}^{(1)}, P^{(1)})$ with chaos expansion (3.1) such that*

$$\|F\|^2_{\mathbb{D}^{(1)}_{1,2}} := \sum_{n\geq 0} nn!\, \|f_n\|^2_{L^2(\lambda^n)} < \infty.$$

Then the Malliavin derivative D_t of $F \in \mathbb{D}^{(1)}_{1,2}$ in the direction of the Brownian motion B is defined as

$$D_t F = \sum_{n\geq 1} n I^{(1)}_{n-1}(\widetilde{f}_{n-1}),$$

where $\widetilde{f}_{n-1}(t_1, \cdots, t_{n-1}) := f_n(t_1, \cdots, t_{n-1}, t)$.

(ii) *Similarly, let $\mathbb{D}^{(2)}_{1,2}$ be the space of all $G \in L^2(\mathcal{F}^{(2)}, P^{(2)})$ with chaos representation (3.2) satisfying*

$$\|G\|^2_{\mathbb{D}^{(2)}_{1,2}} := \sum_{n\geq 0} nn!\, \|g_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty.$$

Then the Malliavin derivative $D_{t,z}$ of $G \in \mathbb{D}^{(2)}_{1,2}$ in the direction of the pure jump Lévy process $\eta_t := \int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(dt, dz)$ is defined as

$$D_{t,z} G := \sum_{n\geq 1} n I^{(2)}_{n-1}(\widetilde{g}_{n-1}),$$

where $\widetilde{g}_{n-1}(t_1, z_1, \cdots, t_{n-1}, z_{n-1}) := g_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t, z)$.

A crucial argument in the proof of our general maximum principle (Theorem 4) rests on duality formulas for the Malliavin derivatives D_t and $D_{t,z}$ [16], [8]:

Lemma 2 (Duality formula for D_t and $D_{t,z}$)

(i) *Require that $\varphi(t)$ is $\mathcal{F}^{(1)}_t$−adapted with $E_{P^{(1)}}\left[\int_0^T \varphi^2(t)\, dt\right] < \infty$ and $F \in \mathbb{D}^{(1)}_{1,2}$. Then*

$$E_{P^{(1)}}\left[F \int_0^T \varphi(t)\, dB(t)\right] = E_{P^{(1)}}\left[\int_0^T \varphi(t) D_t F\, dt\right].$$

(ii) *Assume that* $\psi(t,z)$ *is* $\mathcal{F}_t^{(2)}-$*adapted with* $E_{P^{(2)}}\left[\int_0^T\int_{\mathbb{R}_0}\psi^2(t,z)\,\nu(dz)\,dt\right]<\infty$ *and* $G\in\mathbb{D}_{1,2}^{(2)}$. *Then*

$$E_{P^{(2)}}\left[G\int_0^T\int_{\mathbb{R}_0}\psi(t,z)\widetilde{N}(dt,dz)\right]=E_{P^{(2)}}\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)D_{t,z}G\,\nu(dz)\,dt\right].$$

In the following we shall confine ourselves to the stochastic basis

$$(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},P),$$

where $\Omega=\Omega^{(1)}\times\Omega^{(2)}$, $\mathcal{F}=\mathcal{F}^{(1)}\times\mathcal{F}^{(2)}$, $\mathcal{F}_t=\mathcal{F}_t^{(1)}\times\mathcal{F}_t^{(2)}$, $P=P^{(1)}\times P^{(2)}$.

We remark that we may state the duality relations in Lemma 2 in terms of P.

3.2 Assumptions

In view of the optimization problem (2.4) we require the following conditions 1–5:

1. The functions $b,\ \sigma,\ \theta,\ f,\ g$ are contained in C^1 with respect to the arguments $\Gamma\in\mathbb{R}$ and $u\in U$.
2. For all $0<t\le r<T$ and all bounded $\mathcal{E}_t\otimes\mathcal{B}(\mathbb{R})-$measurable random variables α, the control

$$\beta_\alpha(s,x):=\alpha\cdot\chi_{[t,r]}(s),\ \ 0\le s\le T\,, \tag{3.3}$$

   where $\chi_{[t,T]}$ denotes the indicator function on $[t,T]$, is an admissible control.
3. For all $u,\beta\in\mathcal{A}_{\mathcal{E}}$ with β bounded there exists a $\delta>0$ such that

$$u+y\beta\in\mathcal{A}_{\mathcal{E}} \tag{3.4}$$

   for all $y\in(-\delta,\delta)$,
   and such that the family

$$\Bigg\{\frac{\partial}{\partial\gamma}f(t,x,\Gamma^{u+y\beta}(t,x),u(t,x)+y\beta(t,x),\omega)\frac{d}{dy}\Gamma^{u+y\beta}(t,x)\\+\frac{\partial}{\partial u}f(t,x,\Gamma^{u+y\beta}(t,x),u(t,x)+y\beta(t,x),\omega)\beta(t,x)\Bigg\}_{y\in(-\delta,\delta)}$$

   is $\lambda\times\mathbb{P}\times\mu-$uniformly integrable;

$$\left\{\frac{\partial}{\partial\gamma}g(T,x,\Gamma^{u+y\beta}(T,x),\omega)\frac{d}{dy}\Gamma^{u+y\beta}(T,x)\right\}_{y\in(-\delta,\delta)}$$

   is $\mathbb{P}\times\mu-$uniformly integrable.

4. For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded the process

$$Y(t,x) = Y^{\beta}(t,x) = \left. \frac{d}{dy}\Gamma^{(u+y\beta)}(t,x) \right|_{y=0}$$

exists and

$$\begin{aligned} LY(t,x) &= \left. \frac{d}{dy}L\Gamma^{(u+y\beta)}(t,x) \right|_{y=0} \\ \nabla_x Y(t,x) &= \left. \frac{d}{dy}\nabla_x\Gamma^{(u+y\beta)}(t,x) \right|_{y=0} \end{aligned}$$

Further suppose that $Y(t,x)$ follows the SPDE

$$\begin{aligned} Y(t,x) = &\int_0^t \Big[LY(s,x) + Y(s,x)\frac{\partial}{\partial \gamma}b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega) \\ &+ \nabla_x Y(s,x)\nabla_{\gamma'}b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega)\Big]\, ds \\ &+ \int_0^t \Big[Y(s,x)\frac{\partial}{\partial \gamma}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega) \\ &+ \nabla_x Y(s,x)\nabla_{\gamma'}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega)\Big]\, dB(s) \\ &+ \int_0^t\int_{\mathbb{R}} \Big[Y(s^-,x)\frac{\partial}{\partial \gamma}\theta(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega) \\ &+ \nabla_x Y(s^-,x)\nabla_{\gamma'}\theta(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega)\Big]\ \widetilde{N}(dz,ds) \\ &+ \int_0^t \left[\beta(s,x)\frac{\partial}{\partial u}b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega)\right] ds \\ &+ \int_0^t \beta(s,x)\frac{\partial}{\partial u}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega)\, dB(s) \\ &+ \int_0^t\int_{\mathbb{R}} \beta(s^-,x)\frac{\partial}{\partial u}\theta\,(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega)\ \widetilde{N}(dz,ds)\,, \\ (t,x) \in &[0,T]\times G\,, \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} Y(0,x) =& 0,\ x \in \overline{G}\,, \\ Y(t,x) =& 0,\ (t,x) \in (0,T)\times \partial G\,. \end{aligned}$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$, $\nabla_{\gamma'} = \left(\frac{\partial}{\partial \gamma_1'}, \cdots, \frac{\partial}{\partial \gamma_n'}\right)$ and $\gamma' = \left(\frac{\partial\Gamma}{\partial x_1}, \cdots, \frac{\partial\Gamma}{\partial x_n}\right) = (\gamma_1', \cdots, \gamma_n')$

5. Suppose that for all $u \in \mathcal{A}_{\mathcal{E}}$ the processes

$$
\begin{aligned}
K(t,x) :=& \frac{\partial}{\partial \gamma} g(x, \Gamma(T,x)) + \int_t^T \frac{\partial}{\partial \gamma} f\left(s,x,\Gamma(s,x),u(s,x)\right) ds \\
D_t K(t,x) :=& D_t \frac{\partial}{\partial \gamma} g(x,\Gamma(T,x)) + \int_t^T D_t \left(\frac{\partial}{\partial \gamma} f(s,x,\Gamma(s,x),u(s,x)) \right) ds \\
D_{t,z} K(t,x) :=& D_{t,z} \frac{\partial}{\partial \gamma} g(x,\Gamma(T,x)) + \int_t^T D_{t,z} \left(\frac{\partial}{\partial \gamma} f(s,x,\Gamma(s,x),u(s,x)) \right) ds \\
H_0(s,x,\gamma,\gamma',u) :=& K(s,x) b(s,x,\gamma,\gamma',u,\omega) + D_s K(s,x) \sigma(s,x,\gamma,\gamma',u,\omega) \qquad (3.6) \\
&+ \int_{\mathbb{R}} D_{s,z} K(s,x) \theta(s,x,\gamma,\gamma',u,z,\omega)\, \nu(dz) \\
Z(t,s,x) :=& \exp\left\{ \int_s^t F_{n+1}\left(x, \circ \widehat{dr}\right) \right\}, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
p(t,x) :=& K(t,x) + \int_t^T \left\{ \frac{\partial}{\partial \gamma} H_0(s,x,\Gamma(s,x),\nabla_x \Gamma(s,x),u(s,x)) + L^* K(s,x) \right. \\
&\left. + \nabla_x^* \left(\nabla_{\gamma'} H_0(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x)) \right) \right\} Z(t,s,\varphi_{s,t}(x))\, ds \\
q(t,x) :=& D_t p(t,x) \\
r(t,x,z) :=& D_{t,z} p(t,x); \quad t \in [0,T],\ z \in \mathbb{R}_0,\ x \in G\,.
\end{aligned}
$$

are well-defined and where $\varphi_{s,t}$ and $\varphi_{t,s}^{(i)}$ are defined as before.

Assume also that

$$
\begin{aligned}
& E\left[\int_0^T \int_G \left\{ |K(t,x)| \left(|LY(t,x)| + \left| Y(t,x) \frac{\partial}{\partial \gamma} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \right. \right. \right. \\
&+ \left| \beta(t,x) \frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \\
&+ \left. \left| \nabla_x Y(t,x) \nabla_{\gamma'} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \right) \\
&+ |D_t K(t,x)| \left(\left| Y(t,x) \frac{\partial}{\partial \gamma} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \right. \\
&+ \left| \nabla_x Y(t,x) \nabla_{\gamma'} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \\
&+ \left. \left| \beta(t,x) \frac{\partial}{\partial u} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \right| \right) \\
&+ \int_{\mathbb{R}} |D_{t,z} K(t,x)| \left(\left| Y(t,x) \frac{\partial}{\partial \gamma} \theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega) \right| \right. \\
&+ \left| \nabla_x Y(t,x) \nabla_{\gamma'} \theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega) \right| \\
&+ \left. \left| \beta(t,x) \frac{\partial}{\partial u} \theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega) \right| \right) \nu(dz) \\
&+ \left. \left. \left| \beta(t,x) \frac{\partial}{\partial u} f(t,x,\Gamma(t,x),u(t,x)) \right| \right\} dt\, dx \right] \\
&< \infty.
\end{aligned}
$$

Here L^* is the dual operator of L. Further, the densely defined operator ∇_x^* stands for the adjoint of ∇_x, that is

$$(g, \nabla_x f)_{L^2(G;\mathbb{R}^n)} = (\nabla_x^* g, f)_{L^2(G;\mathbb{R})} \tag{3.8}$$

for all $f \in Dom(\nabla_x), g \in Dom(\nabla_x^*)$. For example, if $g = (g_1, ..., g_n) \in C_0^\infty(G;\mathbb{R}^n)$, then $\nabla_x^* g = \sum\limits_{j=1}^{n} \frac{\partial g_j}{\partial x_j}$.

Let us comment that $D_t K(t,x)$ and $D_{t,z}K(t,x)$ in 5 exist, if e.g. the coefficients b, σ, θ fulfill a global Lipschitz condition, f is independent of u in 1 and the operator L is the generator of a strongly continuous semigroup. See e.g. [16], [19] and [5, Section 5].

3.3 A probabilistic representation of $Y(t,x)$

The proof of our maximum principle (Theorem 4) necessitates a certain probabilistic representation of solutions of the SPDE (3.5). Compare [12] in the Gaussian case. To this end, we need some notations and conditions.

Let $m \in \mathbb{N}$, $0 < \delta \leq 1$. Denote by $C^{m,\delta}$ the space of all m-times continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\|f\|_{m+\delta;K} := \|f\|_{m;K} + \sum_{|\alpha|=m} \sup_{x,y\in K, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\|x-y\|^\delta} < \infty$$

for all compact sets $K \subset \mathbb{R}^n$, where

$$\|f\|_{m;K} := \sup_{x\in K} \frac{|f(x)|}{(1+\|x\|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x\in K} |D^\alpha f(x)|.$$

For the multi-index of non-negative integers $\alpha = (\alpha_1, \cdots, \alpha_d)$ the operator D^α is defined as

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^d)^{\alpha_d}},$$

where $|\alpha| := \sum_{i=1}^d \alpha_i$.

Further introduce for sets $K \subset \mathbb{R}^n$ the norm

$$\|g\|_{m+\delta;K}^\sim := \|g\|_{m;K}^\sim + \sum_{|\alpha|=m} \left\|D_x^\alpha D_y^\alpha g\right\|_{\delta;K}^\sim,$$

where

$$\|g\|_{\delta;K}^\sim := \sup_{\substack{x,y,x',y'\in K\\ x\neq y, x'\neq y'}} \frac{\left|g(x,y) - g(x',y) - g(x,y') + g(x',y')\right|}{\|x-x'\|^\delta \|y-y'\|^\delta}$$

and

$$\|g\|_{m;K}^\sim := \sup_{x,y\in K} \frac{|g(x,y)|}{(1+\|x\|)(1+\|y\|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x,y\in K} \left|D_x^\alpha D_y^\alpha g(x,y)\right|.$$

We shall simply write $\|g\|^{\sim}_{m+\delta}$ for $\|g\|^{\sim}_{m+\delta;\mathbb{R}^n}$.

Define

$$\widetilde{b}_i(t,x) = \frac{\partial}{\partial \gamma_i'} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega), i = 1,\cdots,n \tag{3.9}$$

$$\widetilde{\sigma}_i(t,x) = \frac{\partial}{\partial \gamma_i'} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega), i = 1,\cdots,n \tag{3.10}$$

$$\widetilde{\theta}_i(t,x) = \frac{\partial}{\partial \gamma_i'} \theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega), i = 1,\cdots,n \tag{3.11}$$

$$b^*(t,x) = \frac{\partial}{\partial \gamma} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{3.12}$$

$$\sigma^*(t,x) = \frac{\partial}{\partial \gamma} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{3.13}$$

$$\theta^*(t,x,z) = \frac{\partial}{\partial \gamma} \theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega) \tag{3.14}$$

$$b_u(t,x) := \beta(s,x)\frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{3.15}$$

$$\sigma_u(t,x) := \beta(s,x)\frac{\partial}{\partial u} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{3.16}$$

Set

$$\begin{aligned}
F_i(x,dt) &:= \widetilde{b}_i(t,x)\,dt + \widetilde{\sigma}_i(t,x)\,dB(t), \quad i = 1,\cdots,n \\
F_{n+1}(x,dt) &:= b^*(t,x)\,dt + \sigma^*(t,x)\,dB(t) + \int_{\mathbb{R}_0} \theta^*(t,x,z)\widetilde{N}(dt,dz) \\
F_{n+2}(x,t) &:= \int_0^t b_u(s,x)\,ds + \int_0^t \sigma_u(s,x)\,dB(s)
\end{aligned}$$

Define the symmetric matrix function $(A^{ij}(x,y,s)_{1\le i,j\le n+2}$ given by

$$\begin{aligned}
A^{ij}(x,y,s) &= \widetilde{\sigma}_i(s,x)\cdot\widetilde{\sigma}_j(s,y), \quad i,j = 1,\cdots,n, \\
A^{i,n+1}(x,y,s) &= \widetilde{\sigma}_i(s,x)\sigma^*(s,y), \quad i = 1,\cdots,n \\
A^{i,n+2}(x,y,s) &= \widetilde{\sigma}_i(s,x)\sigma_u(s,y), \quad i = 1,\cdots,n
\end{aligned}$$

and

$$\begin{aligned}
A^{n+1,n+1}(x,y,s) &= \sigma^*(s,x)\cdot\sigma^*(s,y) \\
A^{n+1,n+2}(x,y,s) &= \sigma^*(s,x)\cdot\sigma_u(s,y) \\
A^{n+2,n+2}(x,y,s) &= \sigma_u(s,x)\cdot\sigma_u(s,y)
\end{aligned}$$

We make the following assumptions:

D1 $\frac{\partial}{\partial u}\theta(t,x,\Gamma(t,x), \;\; \nabla_x\Gamma(t,x),u(t,x),z,\omega) \equiv 0, \;\; \widetilde{\theta}_i(t,x) \equiv 0, \;\; i = 1,\cdots,n.$

D2 $\sigma^*(t,x), \theta^*(t,x,z), \;\; \widetilde{\sigma}_i(t,x), \;\; i = 1,\cdots,n$ are measurable deterministic functions.

D3 $\sum_{i,j=1}^{n+2}\int_0^T \left\|A^{ij}(\cdot,\cdot,s)\right\|_{m+\delta}^{\sim} ds < \infty$ and
$\int_0^T \left\{ (\sum_{i=1}^n \left\|\widehat{b}_i(s,\cdot)\right\|_{m+\delta}) + \|b^*(s,\cdot)\|_{m+\delta} + \|b_u(s,\cdot)\|_{m+\delta} \right\} ds < \infty$ a.e.
for some $m \geq 3$ and $\delta > 0$.

D4 There exists a measurable function $(z \longmapsto \beta(r,z))$ such that

$$\left|D_x^\alpha \theta^*(t,x,z) - D_x^\alpha \theta^*(t,x^{'},z)\right| \leq \beta(r,z)\left\|x - x^{'}\right\|^\delta$$

and

$$\int_{\mathbb{R}_0} |\beta(r,z)|^p \,\nu(dz) < \infty$$

for all $p \geq 2, |\alpha| \leq 2, 0 \leq t \leq T$ and $x, x^{'}$ with $\|x\| \leq r, \left\|x^{'}\right\| \leq r$.

D5 There exist measurable functions $\alpha(z) \leq 0 \leq \beta(z)$ such that

$$-1 < \alpha(z) \leq \theta^*(t,x,z) \leq \beta(z) \text{ for all } t,x,z$$

and

$$\int_{\mathbb{R}_0} |\beta(z)|^p \,\nu(dz) + \int_{\mathbb{R}_0} (\alpha(z) - \log(1+\alpha(z)))^{p/2} \,\nu(dz) < \infty \text{ for all } p \geq 2.$$

In the following we assume that the differential operator L in Equation (3.5) is of the form

$$L_s u = L_s^{(1)} u + L_s^{(2)} u,$$

where

$$L_s^{(1)} u := \frac{1}{2}\sum_{i,j=1}^n a^{ij}(x,s)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n b_i(x,s)\frac{\partial u}{\partial x^i} + d(x,s)u$$

and

$$\begin{aligned} L_s^{(2)} u \quad &:= \quad \frac{1}{2}\sum_{i,j=1}^n A^{ij}(x,x,s)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n \left(A^{i,n+1}(x,x,s) + \frac{1}{2}C_i(x,s)\right)\frac{\partial u}{\partial x^i} \\ &\quad + \frac{1}{2}\left(D(x,s) + A^{n+1,n+1}(x,x,s)\right)u \end{aligned}$$

where $d(x,s)$ is a function that fulfills condition D9 below and

$$C_j(x,s) := \sum_{j=1}^n \frac{\partial A^{ij}}{\partial y^i}(x,y,s)\bigg|_{y=x}, \quad i = 1,\cdots,n\,,$$

$$D(x,s) := \sum_{j=1}^n \frac{\partial A^{i,n+1}}{\partial y^i}(x,y,s)\bigg|_{y=x}.$$

We require the following conditions:

D6 $L_t^{(1)}$ is an elliptic differential operator.

D7 There exists a non-negative symmetric continuous matrix function $(a^{ij}(x,y,s))_{1\le i,j\le n}$ such that $a^{ij}(x,x,s) = a^{ij}(x,s)$. Further it is assumed that

$$\sum_{i,j=1}^{n} \left\| a^{ij}(\cdot,s) \right\|_{m+1+\delta} \le K \text{ for all } s$$

for a constant K and some $m \ge 3$, $\delta > 0$.

D8 The functions $b_i(x,s), i = 1,\cdots,n$ are continuous in (x,s) and satisfy

$$\sum_{i=1}^{n} \|b_i(\cdot,s)\|_{m+\delta} \le C \text{ for all } s$$

for a constant C and some $m \ge 3$, $\delta > 0$.

D9 The function $d(x,s)$ is continuous in (x,s) and belongs to $C^{m,\delta}$ for some $m \ge 3$, $\delta > 0$. In addition a^{ij} is bounded and $d/(1+\|x\|)$ is bounded from the above.

D10 The functions b^*, σ^* and d^* are uniformly bounded.

We now derive the announced representation of a solution $Y(t,x)$ of Equation (3.5). Let $X(x,t) = (X_1(x,t),\cdots,X_n(x,t))$ be a $C^{k,\gamma}-$valued Brownian motion, that is a continuous process $X(t,\cdot) \in C^{k,\gamma}$ with independent increments (see [12]) on another probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$. Assume that this process has local characteristic $a^{ij}(x,y,t)$ and $m(x,t) = b(x,t) - c(x,t)$, where the correction term $c(x,t)$ is given by

$$c_i(x,t) = \frac{1}{2}\int_0^t \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x^j}(x,y,s)\bigg|_{y=x} ds, \ \ i = 1,\cdots,n.$$

Then, let us consider on the product space $(\Omega\times\widehat{\Omega}, \mathcal{F}\times\widehat{\mathcal{F}}, P\times\widehat{P})$ the first order SPDE

$$\begin{aligned} v(x,t) = &\sum_{i=1}^{n}\int_0^t (X_i(x,\circ ds) + F_i(x,\circ ds))\frac{\partial v(x,s)}{\partial x^i} \\ &+ \int_0^t (d(x,s)ds + F_{n+1}(x,\circ ds))v(x,s) + F_{n+2}(x,t), \end{aligned} \tag{3.17}$$

where $\circ dt$ stands for non-linear integration in the sense of Stratonovich (see [12]). Using the definition of $X(x,t)$ the equation (3.17) can be recast as

$$\begin{aligned} v(x,t) = &\int_0^t L_s v(x,s)ds + \sum_{i=1}^{n} Y_i^*(x,ds)\frac{\partial v(x,s)}{\partial x^i} \\ &+ \sum_{i=1}^{n}\int_0^t F_i(x,ds)\frac{\partial v(x,s)}{\partial x^i} + \int_0^t F_{n+1}(x,ds)v(x,s) \\ &+ F_{n+2}(x,t), \end{aligned} \tag{3.18}$$

where $Y^*(x,t) = (Y_1^*(x,t), ..., Y_n^*(x,t))$ is the martingale part of $X(t,x)$. So applying the expectation $E_{\widehat{P}}$ to both sides of the latter equation gives the following representation for the solution to (3.5) (See also the proof of Theorem 6.2.5 in [12]):

Proposition 3 *Under the above specified conditions we obtain the following probabilistic representation*

$$Y(t,x) = E_{\widehat{P}}\left[v(x,t)\right] . \tag{3.19}$$

In order to use representation (3.19) in the proof of our general stochastic maximum principle for SPDE's (Theorem 3) we proceed to develop an expression for $v(x,t)$. Let $\varphi_{s,t}$ be the solution of the Stratonovich SDE

$$\varphi_{s,t}(x) = x - \int_s^t G(\varphi_{s,r}(x), \circ dr),$$

where $G(x,t) := (X_1(x,t) + F_1(x,t), \cdots, X_n(x,t) + F_n(x,t))$. Then by employing the proof of Theorem 6.1.8 and Theorem 6.1.9 in [12] with respect to a generalized Itô formula in [4] one obtains the following representation of $v(t,x)$:

$$\begin{aligned}
&v(x,t) \\
&= \int_0^t \exp\left\{\frac{1}{2}\int_s^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_s^t b^*(r,\varphi_{t,r}(x))\,dr + \sigma^*(r,\varphi_{t,r}(x))\,\widehat{d}B(r)\right. \\
&\quad + \int_s^t \int_{\mathbb{R}_0} (\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z))\,dr \\
&\quad \left. + \int_s^t \int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\,\widetilde{N}(\widehat{d}r,dz)\right\} \times \\
&\quad \left(\beta(s,x)\frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\,ds\right. \\
&\quad \left. +\beta(s,x)\frac{\partial}{\partial u}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\circ\widehat{d}B(s)\right),
\end{aligned} \tag{3.20}$$

where $\widehat{d}$ denotes backward integration and where the inverse flow $\varphi_{t,s} = \varphi_{s,t}^{-1}$ solves the backward Stratonovich SDE

$$\varphi_{t,s}^{(i)}(x) = x_i + \int_s^t \widetilde{b}_i(r,\varphi_{t,r}(x))\,dr + \int_s^t \widetilde{\sigma}_i(r,\varphi_{t,r}(x)) \circ \widehat{d}B(r), \;\; i = 1, ..., n.$$

For later use, we end this subsection to consider the case with general boundary condition $f(x)$, that is

$$\begin{aligned}
Y(0,x) &= f(x),\ x \in \overline{G}\,, \\
Y(t,x) &= 0,\ \ (t,x) \in (0,T) \times \partial G\,,
\end{aligned}$$

holds, where $f \in C^{m,\delta}$.

Then, $v(x,t)$ is described by

$$
\begin{aligned}
v(x,t) =& f(x) + \int_0^t L_s v(x,s) ds + \sum_{i=1}^n Y_i^*(x,ds) \frac{\partial v(x,s)}{\partial x^i} \\
&+ \sum_{i=1}^n \int_0^t F_i(x,ds) \frac{\partial v(x,s)}{\partial x^i} + \int_0^t F_{n+1}(x,ds) v(x,s) \\
&+ F_{n+2}(x,t)\,,
\end{aligned}
$$

and using the same reasoning as above we obtain:

$$
\begin{aligned}
& v(x,t) \\
=& \exp\left\{ \frac{1}{2} \int_0^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_0^t b^*(r,\varphi_{t,r}(x))\, dr + \sigma^*(r,\varphi_{t,r}(x))\, \widehat{d}B(r) \right. \\
&+ \int_0^t \int_{\mathbb{R}_0} (\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z))\, dr \\
&+ \left. \int_0^t \int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\, \widetilde{N}(\widehat{d}r,dz) \right\} \times f(\varphi_{t,0}(x)) \\
&+ \int_0^t \exp\left\{ \frac{1}{2} \int_s^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_s^t b^*(r,\varphi_{t,r}(x))\, dr + \sigma^*(r,\varphi_{t,r}(x))\, \widehat{d}B(r) \right. \\
&+ \int_s^t \int_{\mathbb{R}_0} (\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z))\, dr \\
&+ \left. \int_s^t \int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\, \widetilde{N}(\widehat{d}r,dz) \right\} \times \\
&\Bigg(\beta(s,x) \frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\, ds \\
&+ \beta(s,x) \frac{\partial}{\partial u} \sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \circ \widehat{d}B(s) \Bigg)\,, \qquad (3.21)
\end{aligned}
$$

3.4 A general stochastic maximum principle for a partial information control problem

We are now ready to state a general stochastic maximum principle for our partial information control problem (2.4). To this end we introduce the *general Hamiltonian*

$$
H : [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R}
$$

by

$$
\begin{aligned}
H(t,x,\gamma,\gamma',u,\omega) \quad := \quad & f(t,x,\gamma,u,\omega) + p(t,x) b(t,x,\gamma,\gamma',u,\omega) + D_t p(t,x) \sigma(t,x,\gamma,\gamma',u,\omega) \\
& + \int_{\mathbb{R}} D_{t,z} p(t,x) \theta(t,x,\gamma,\gamma',u,z,\omega)\, \nu(dz). \qquad (3.22)
\end{aligned}
$$

We then have

Theorem 4 *Retain the conditions 1-5. Assume that $\widehat{u} \in \mathcal{A}_{\mathcal{E}}$ is a* critical point *of the performance functional $J(u)$ in (2.4), that is*

$$\frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} = 0 \tag{3.23}$$

for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$. Then

$$E\left[E_Q\left[\int_G \frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))dx\right]\Big|\mathcal{E}_t\right] = 0 \quad a.e. \ in \quad (t,x,\omega), \tag{3.24}$$

where

$$\begin{aligned}
\widehat{\Gamma}(t,x) &= \Gamma^{(\widehat{u})}(t,x),\\
\widehat{H}(t,x,\gamma,\gamma',u,\omega) &= f(t,x,\gamma,u,\omega) + \widehat{p}(t,x)b(t,x,\gamma,\gamma',u,\omega) + D_t\widehat{p}(t,x)\sigma(t,x,\gamma,\gamma',u,\omega)\\
&\quad + \int_{\mathbb{R}} D_{t,z}\widehat{p}(t,x)\theta(t,x,\gamma,\gamma',u,z,\omega)\nu(dz)\,,
\end{aligned}$$

with

$$\begin{aligned}
\widehat{p}(t,x) &= \widehat{K}(t,x) + \int_t^T \Big\{\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x,\widehat{\Gamma},\widehat{\Gamma}',\widehat{u},\omega) + L^*\widehat{K}(s,x)\\
&\quad + \nabla_x^*\Big(\nabla_{\gamma'}H_0(s,x,\widehat{\Gamma},\widehat{\Gamma}',\widehat{u},\omega)\Big)\Big\}\,\widehat{Z}(t,s,\widehat{\varphi}_{s,t}(x))ds\,,
\end{aligned}$$

$$\widehat{K}(t,x) = \frac{\partial}{\partial\gamma}g(x,\widehat{\Gamma}(T,x),\omega) + \int_t^T \frac{\partial}{\partial\gamma}f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)ds,$$

where $\widehat{\varphi}_{s,t}$ is the solution of the of the Stratonovich SDE

$$\varphi_{s,t}(x) = x - \int_s^t \widehat{G}(\widehat{\varphi}_{s,r}(x),\circ dr),$$

with $\widehat{G}(x,t) := (X_1(x,t) + \widehat{F}_1(x,t),\cdots,X_n(x,t) + \widehat{F}_n(x,t))$,

$$\begin{aligned}
\widehat{\varphi}_{s,t}(x) &= x - \int_s^t \widehat{G}(\varphi_{s,r}(x),\circ dr),\\
\widehat{F}_i(x,dt) &:= \widehat{\widehat{b}}_i(t,x)\,dt + \widehat{\widehat{\sigma}}_i(t,x)\,dB(t), \quad i = 1,\cdots,n\\
\widehat{F}_{n+1}(x,dt) &:= \widehat{b}^*(t,x)\,dt + \widehat{\sigma}^*(t,x)\,dB(t) + \int_{\mathbb{R}_0}\widehat{\theta}^*(t,x,z)\widetilde{N}(dt,dz)\\
\widehat{F}_{n+2}(x,t) &:= \int_0^t \widehat{b}_u(s,x)\,ds + \int_0^t \widehat{\sigma}_u(s,x)\,dB(s)
\end{aligned}$$

$$\widehat{\widehat{b}}_i(t,x) = \frac{\partial}{\partial\gamma_i'}b(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega), \quad i = 1,\cdots,n$$

$$\widehat{\widehat{\sigma}}_i(t,x) = \frac{\partial}{\partial \gamma_i'}\sigma(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega), \quad i=1,\cdots,n$$

$$\widehat{\widehat{\theta}}_i(t,x) = \frac{\partial}{\partial \gamma_i'}\theta(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),z,\omega), \quad i=1,\cdots,n$$

$$\widehat{b}^*(t,x) = \frac{\partial}{\partial \gamma}b(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega)$$

$$\widehat{\sigma}^*(t,x) = \frac{\partial}{\partial \gamma}\sigma(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega)$$

$$\widehat{\theta}^*(t,x,z) = \frac{\partial}{\partial \gamma}\theta(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),z,\omega)$$

$$\widehat{b}_u(t,x) := \beta(s,x)\frac{\partial}{\partial u}b(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega)$$

$$\widehat{\sigma}_u(t,x) := \beta(s,x)\frac{\partial}{\partial u}\sigma(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega),$$

and

$$\widehat{Z}(t,s) := \exp\left\{\int_s^t F_{n+1}\left(\widehat{\varphi}_{s,r}(x), \circ\widehat{dr}\right)\right\},$$

Remark 5 *We remark that in Theorem 4 the partial derivatives of H and H_0 with respect to u, γ, and γ' only refer to differentiation at places where the arguments appear in the coefficients of the definitions* (3.6) *and* (3.22)*.*

Proof. Since $\widehat{u} \in \mathcal{A}_{\mathcal{E}}$ is a critical point, there exists for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ a $\delta > 0$ as in (3.4). We conclude that

$$\begin{aligned} 0 \quad = \quad & \frac{d}{dy}J(\widehat{u}+y\beta)\Big|_{y=0} = E\left[\int_0^T \int_G \left(\frac{\partial}{\partial \gamma}f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)\widehat{Y}^{\beta}(s,x)\right.\right. \\ & \left.\left. +\frac{\partial}{\partial u}f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)\beta(s,x)\right) dx\, ds + \int_G \frac{\partial}{\partial \gamma}g(x,\widehat{\Gamma}(T,x),\omega)\widehat{Y}^{\beta}(T,x)\, dx\right], \end{aligned}$$

where $\widehat{Y}^\beta$ is defined as in 4 with $u = \widehat{u}$ and fulfills

$$\begin{aligned}
\widehat{Y}^\beta(t,x) = & \int_0^t \Big[L\widehat{Y}^\beta(s,x) + \widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma} b(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(s,x), \widehat{u}(s,x)) \\
& + \nabla_x \widehat{Y}^\beta(s,x)\nabla_{\gamma'} b(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(s,x), \widehat{u}(s,x))\Big]\, ds \\
& + \int_0^t \Big[\widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma}\sigma(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x)) \\
& + \nabla_x \widehat{Y}^\beta(s,x)\nabla_{\gamma'}\sigma(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x))\Big]\, dB(s) \\
& + \int_0^t \int_{\mathbb{R}} \Big[\widehat{Y}^\beta(s^-,x)\frac{\partial}{\partial\gamma}\theta(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x), z) \\
& + \nabla_x \widehat{Y}^\beta(s^-,x)\nabla_{\gamma'}\theta(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x), z)\Big]\, \widetilde{N}(dz,ds) \\
& + \int_0^t \Big[\beta(s,x)\frac{\partial}{\partial u} b(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(s,x), \widehat{u}(s,x))\Big]\, ds \\
& + \int_0^t \beta(s,x)\frac{\partial}{\partial u}\sigma(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x))\, dB(s) \\
& + \int_0^t \int_{\mathbb{R}} \beta(s^-,x)\frac{\partial}{\partial u}\theta(s,x,\widehat{\Gamma}(s,x), \nabla_x\widehat{\Gamma}(t,x), \widehat{u}(s,x), z)\, \widetilde{N}(dz,ds) \qquad (3.25)\\
(t,x) \in [0,T]\times G &
\end{aligned}$$

with

$$\begin{aligned}
\widehat{Y}^\beta(0,x) &= 0,\ x \in \overline{G} \\
\widehat{Y}^\beta(t,x) &= 0,\ (t,x) \in (0,T)\times\partial G.
\end{aligned}$$

Using the short hand notation $\frac{\partial}{\partial\gamma} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega) = \frac{\partial}{\partial\gamma} f(s,x)$, $\frac{\partial}{\partial u} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega) = \frac{\partial}{\partial u} f(s,x)$ and similarly for $\frac{\partial g}{\partial\gamma}$, $\frac{\partial b}{\partial\gamma}$, $\frac{\partial b}{\partial u}$, $\frac{\partial\sigma}{\partial\gamma}$, $\frac{\partial\sigma}{\partial u}$, $\frac{\partial\theta}{\partial\gamma}$ and $\frac{\partial\theta}{\partial u}$, we can write

$$\begin{aligned}
& E\left[\int_G \frac{\partial}{\partial\gamma} g(x,\widehat{\Gamma}(T,x))\widehat{Y}^\beta(T,x)dx\right] \\
= & \int_G E\left[\frac{\partial}{\partial\gamma} g(x,\widehat{\Gamma}(T,x))\widehat{Y}^\beta(T,x)\right] dx \\
= & \int_G E\Bigg[\frac{\partial}{\partial\gamma} g(x,\widehat{\Gamma}(T,x))\Bigg(\int_0^T \Big[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial\gamma} b(t,x)\widehat{Y}^\beta(t,x) \\
& + \nabla_x\widehat{Y}^\beta(t,x)\frac{\partial}{\partial\gamma'} b(t,x) + \beta(t,x)\frac{\partial}{\partial u} b(t,x)\Big]\, dt \\
& + \int_0^T \left[\frac{\partial}{\partial\gamma}\sigma(t,x,)\widehat{Y}^\beta(t,x) + \nabla_x\widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma'}\sigma(t,x) + \frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\right] dB(t) \\
& + \int_0^T \int_{\mathbb{R}_0} \left[\frac{\partial}{\partial\gamma}\theta(t,x,z) + \nabla_x\widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma'}\theta(t,x,z) + \frac{\partial}{\partial u}\theta(t,x,z)\beta(t^-,x)\right]\widetilde{N}(dt,dz)\Bigg)\Bigg] dx
\end{aligned}$$

Then by the duality formulas (Lemma 2) we get that

$$
\begin{aligned}
& E\left[\int_G \frac{\partial}{\partial \gamma} g(x,\widehat{\Gamma}(T,x))\widehat{Y}^\beta(T,x)\,dx\right] \\
= \ & \int_G E\left[\int_0^T \left(\frac{\partial}{\partial \gamma} g(x,\widehat{\Gamma}(T,x))\left[L\widehat{Y}^\beta(t,x)+\frac{\partial}{\partial \gamma}b(t,x)\widehat{Y}^\beta(t,x)\right.\right.\right. \\
& \left.+\nabla_{\gamma'}b(t,x)\nabla_x Y(t,x)+\frac{\partial}{\partial u}b(t,x)\beta(t,x)\right] \\
& +D_t\left(\frac{\partial}{\partial \gamma} g(x,\widehat{\Gamma}(T,x))\right)\left[\frac{\partial}{\partial \gamma}\sigma(t,x)\widehat{Y}^\beta(t,x)+\nabla_{\gamma'}\sigma(t,x)\nabla_x\widehat{Y}^\beta(t,x)\right. \\
& \left.+\frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\right]+\int_{\mathbb{R}_0}\left\{D_{t,z}\left(\frac{\partial}{\partial \gamma} g(x,\widehat{\Gamma}(T,x))\right)\left[\frac{\partial}{\partial \gamma}\theta(t,x,z)\widehat{Y}^\beta(t^-,x)\right.\right. \\
& \left.\left.\left.\left.+\nabla_{\gamma'}\theta(t,x,z)\nabla_x\widehat{Y}^\beta(t^-,x)+\frac{\partial}{\partial u}\theta(t,x,z)\beta(t^-,x)\right]\right\}\nu(dz)\right)dt\right]dx.
\end{aligned}
\tag{3.26}
$$

Further we similarly obtain by duality and Fubini's theorem that

$$
\begin{aligned}
& E\left[\int_0^T\int_G \frac{\partial}{\partial \gamma} f(t,x)\widehat{Y}^\beta(t,x)\,dx\,dt\right] \\
= \ & E\left[\int_0^T\int_G \frac{\partial}{\partial \gamma} f(t,x)\left(\int_0^t\left\{L\widehat{Y}^\beta(s,x)+\frac{\partial}{\partial \gamma}b(s,x)\widehat{Y}^\beta(s,x)\right.\right.\right. \\
& \left.+\frac{\partial}{\partial u}b(s,x)\beta(s,x)+\nabla_{\gamma'}b(s,x,)\nabla_x Y(s,x)\right\}ds \\
& +\int_0^t\left\{\frac{\partial}{\partial \gamma}\sigma(s,x)\widehat{Y}^\beta(s,x)+\nabla_{\gamma'}\sigma(s,x)\nabla_x\widehat{Y}^\beta(s,x)+\frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\right\}dB(s) \\
& \left.\left.+\int_0^t\left\{\frac{\partial}{\partial \gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x)+\nabla_{\gamma'}\theta(s,x,z)\nabla_x\widehat{Y}^\beta(t,x)+\frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\right\}\widetilde{N}(dz,ds)\right)dxdt\right] \\
= \ & \int_G E\left[\int_0^T\left(\int_0^t\left\{\frac{\partial}{\partial \gamma} f(t,x)\left[L\widehat{Y}^\beta(s,x)+\frac{\partial}{\partial \gamma}b(s,x)\widehat{Y}^\beta(s,x)+\nabla_{\gamma'}b(s,x,)\nabla_x Y(s,x)\right.\right.\right.\right. \\
& \left.+\frac{\partial}{\partial u}b(s,x)\beta(s,x)\right]+D_s\left(\frac{\partial}{\partial \gamma} f(t,x)\right)\left[\frac{\partial}{\partial \gamma}\sigma(s,x)\widehat{Y}^\beta(s,x)+\nabla_{\gamma'}\sigma(s,x)\nabla_x\widehat{Y}^\beta(s,x)\right. \\
& \left.+\frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\right]+\int_{\mathbb{R}_0}D_{s,z}\left(\frac{\partial}{\partial \gamma} f(t,x)\right)\left[\frac{\partial}{\partial \gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x)\right. \\
& \left.\left.\left.\left.+\nabla_{\gamma'}\theta(s,x,z)\nabla_x\widehat{Y}^\beta(t,x)+\frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\right]\nu(dz)\right\}ds\right)dt\right]dx
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \int_G E\Bigg[\int_0^T \Bigg\{\Bigg(\int_s^T \frac{\partial}{\partial\gamma} f(t,x)dt\Bigg)\Bigg[L\widehat{Y}^\beta(s,x) + \frac{\partial}{\partial\gamma} b(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'} b(s,x,)\nabla_x Y(s,x) \\
& + \frac{\partial}{\partial u} b(s,x)\beta(s,x)\Bigg] + \Bigg(\int_s^T D_s \frac{\partial}{\partial\gamma} f(t,x)dt\Bigg)\Bigg[\frac{\partial}{\partial\gamma}\sigma(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'}\sigma(s,x)\nabla_x\widehat{Y}^\beta(s,x) \\
& + \frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\Bigg] + \int_{\mathbb{R}_0}\Bigg(\int_s^T D_{s,z}\frac{\partial}{\partial\gamma} f(t,x)dt\Bigg)\Bigg[\frac{\partial}{\partial\gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x) \\
& + \nabla_{\gamma'}\theta(s,x,z)\nabla_x\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\Bigg]\nu(dz)\Bigg\} ds\Bigg] dx
\end{aligned}
$$

Changing the notation $s \to t$, this becomes

$$
\begin{aligned}
= \quad & \int_G E\Bigg[\int_0^T \Bigg\{\Bigg(\int_t^T \frac{\partial}{\partial\gamma} f(s,x)ds\Bigg)\Bigg[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial\gamma} b(t,x)\widehat{Y}^\beta(t,x) \\
& + \frac{\partial}{\partial u} b(t,x)\beta(t,x) + \nabla_{\gamma'} b(t,x,)\nabla_x Y(t,x)\Bigg] \\
& + \Bigg(\int_t^T D_t \frac{\partial}{\partial\gamma} f(s,x)ds\Bigg)\Bigg[\frac{\partial}{\partial\gamma}\sigma(t,x)\widehat{Y}^\beta(t,x) + \nabla_{\gamma'}\sigma(t,x)\nabla_x\widehat{Y}^\beta(t,x) \\
& + \frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\Bigg] + \int_{\mathbb{R}_0}\Bigg(\int_t^T D_{t,z}\frac{\partial}{\partial\gamma} f(s,x)ds\Bigg)\Bigg[\frac{\partial}{\partial\gamma}\theta(t,x,z)\widehat{Y}^\beta(t,x) \\
& + \nabla_{\gamma'}\theta(t,x,z)\nabla_x\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(t,x,z)\beta(t,x)\Bigg]\nu(dz)\Bigg\} dt\Bigg] dx \qquad (3.27)
\end{aligned}
$$

Thus by the definition of $\widehat{K}(t,x)$ and combining with (3.23)-(3.27) it follows that

$$
\begin{aligned}
& E\Bigg[\int_G\int_0^T \Bigg\{\widehat{K}(t,x)\Bigg[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial\gamma} b(t,x)\widehat{Y}^\beta(t,x) + \nabla_{\gamma'} b(t,x)\nabla_x\widehat{Y}^\beta(t,x) \\
& + \frac{\partial}{\partial u} b(t,x)\beta(t,x)\Bigg] + D_t\widehat{K}(t,x)\Bigg[\frac{\partial}{\partial\gamma}\sigma(t,x)\widehat{Y}^\beta(t,x) + \nabla_{\gamma'}\sigma(t,x)\widehat{Y}^\beta(t,x) \\
& + \frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\Bigg] + \int_{\mathbb{R}_0}\Bigg\{D_{t,z}\widehat{K}(t,x)\Bigg[\frac{\partial}{\partial\gamma}\theta(t,x,z)\widehat{Y}^\beta(r,x) \\
& + \nabla_{\gamma'}\theta(t,x,z)\nabla_x\widehat{Y}^\beta(r,x) + \frac{\partial}{\partial u}\theta(t,x,z)\beta(t,x)\Bigg]\Bigg\}\nu(dz) \\
& + \frac{\partial}{\partial u} f(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega)\beta(t,x)\Bigg\} dtdx\Bigg] = 0 \qquad (3.28)
\end{aligned}
$$

We observe that for all $\beta = \beta_\alpha \in \mathcal{A}_\mathcal{E}$ of the form $\beta_\alpha(s,x) = \alpha\chi_{[t,t+h]}(s)$ for some $t,h \in (0,T)$, $t+h \le T$ as defined in (3.3)

$$
\widehat{Y}^{\beta_\alpha}(s,x) = 0, \ 0 \le s \le t, x \in G\,.
$$

Then by inspecting (3.28) we have that

$$
A_1 + A_2 + A_3 + A_4 = 0 \qquad (3.29)
$$

where

$$
\begin{aligned}
A_1 &= E\Bigg[\int_G \int_t^T \Big\{\widehat{K}(s,x)\frac{\partial}{\partial\gamma}b(s,x) + D_s\widehat{K}(s,x)\frac{\partial}{\partial\gamma}\sigma(s,x) \\
&\quad + \int_{\mathbb{R}} D_{s,z}\widehat{K}(s,x)\frac{\partial}{\partial\gamma}\theta(s,x,z)\nu(dz)\Big\}\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\Bigg] \\
A_2 &= E\Bigg[\int_G \int_t^{t+h} \Big\{\widehat{K}(s,x)\frac{\partial}{\partial u}b(s,x) + D_s\widehat{K}(s,x)\frac{\partial}{\partial u}\sigma(s,x) \\
&\quad + \int_{\mathbb{R}} D_{s,z}\widehat{K}(s,x)\frac{\partial}{\partial u}\theta(s,x,z)\nu(dz) + \frac{\partial}{\partial u}f(s,x)\Big\}\alpha\,ds\,dx\Bigg] \\
A_3 &= E\Bigg[\int_G \int_t^T \widehat{K}(s,x)L\widehat{Y}^{\beta_\alpha}(s,x)\,dx\,dt\Bigg] \\
A_4 &= E\Bigg[\int_G \int_t^T \Big\{\widehat{K}(s,x)\frac{\partial}{\partial\gamma'}b(s,x) + D_s\widehat{K}(s,x)\nabla_{\gamma'}\sigma(s,x) \\
&\quad + \int_{\mathbb{R}} D_{s,z}\widehat{K}(s,x)\nabla_{\gamma'}\theta(s,x,z)\,\nu(dz)\Big\}\nabla_x\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\Bigg]
\end{aligned}
$$

Note by the definition of $\widehat{Y}^{\beta_\alpha}$ with $\widehat{Y}^{\beta_\alpha}(s,x) = Y(s,x)$ and $s \geq t+h$ the process $Y(s,x)$ follows the following SPDE

$$
\begin{aligned}
d\widehat{Y}(s,x) = &\Big\{L\widehat{Y}^{\beta_\alpha}(s,x) + \widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}b(s,x) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}b(s,x)\Big\}\,ds \\
&+ \Big\{\widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}\sigma(s,x) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}\sigma(s,x)\Big\}\,dB(s) \\
&+ \int_{\mathbb{R}_0}\Big\{\widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}\theta(s,x,z) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}\theta(s,x,z)\Big\}\,\widetilde{N}(dz,dr)
\end{aligned}
$$

Using notation (3.9)-(3.16) and assumption D1 we have

$$
\begin{aligned}
d\widehat{Y}(s,x) =& L\widehat{Y}^{\beta_\alpha}(s,x) + \widehat{Y}^{\beta_\alpha}(s^-,x)\Big\{b^*(s,x)\,ds + \sigma^*(s,x)\,dB(s) + \int_{\mathbb{R}_0}\theta^*(s,x,z)\,\widetilde{N}(dz,dr)\Big\} \\
&+ \sum_{i=1}^n \frac{\partial}{\partial x_i}\widehat{Y}^{\beta_\alpha}(s^-,x)\Big\{\widetilde{b}_i(s,x)\,ds + \widetilde{\sigma}_i(s,x)\,dB(s)\Big\}, \qquad (3.30)
\end{aligned}
$$

for $s \geq t+h$ with initial condition $Y(t+h,x) \neq 0$ at time $t+h$. Equation (3.30) can be solved explicitly using the stochastic flow theory of the preceding section.

Let us consider the equation (see p. 297/298 in [12])

$$
\eta_s(y) = \int_0^s \eta_r(y)F_{n+1}(\varphi_{0,r}(x),\circ dr) + \int_0^s F_{n+2}(\varphi_{0,r}(x),\circ dr).
$$

Then

$$\begin{aligned}\eta_s(y) = &\int_0^{t+h} \eta_r(y) F_{n+1}(\varphi_{0,r}(x), \circ dr) + \int_0^{t+h} F_{n+2}(\varphi_{0,r}(x), \circ dr) \\ &+ \int_{t+h}^{s} \eta_r(y) F_{n+1}(\varphi_{0,r}(x), \circ dr) + \int_{t+h}^{s} F_{n+2}(\varphi_{0,r}(x), \circ dr) \\ =&\eta_{t+h}(y) + \int_{t+h}^{s} \eta_r(y) F_{n+1}(\varphi_{0,r}(x), \circ dr) + 0 \,.\end{aligned}$$

So it follows that

$$\eta_s(y) = \eta_{t+h}(y) \exp\left\{ \int_{t+h}^{s} F_{n+1}(\varphi_{0,r}(x), \circ dr) \right\}.$$

Thus, from (3.21) we derive

$$\begin{aligned} v(x,s) &= \eta_s(y)|_{y=\varphi_{s,0}(x)} = \eta_{t+h}(y)|_{y=\varphi_{s,0}(x)} \exp\left\{ \int_{t+h}^{s} F_{n+1}(\varphi_{s,r}(x), \circ \widehat{dr}) \right\} \\ &= v(\varphi_{s,t+h}(x), t+h) \exp\left\{ \int_{t+h}^{s} F_{n+1}(\varphi_{s,r}(x), \circ \widehat{dr}) \right\}. \end{aligned} \tag{3.31}$$

Therefore, using representation (3.19) together with (3.31), we obtain that

$$\begin{aligned} Y(s,x) =&E_Q\left[v(\varphi_{s,t+h}(x), t+h) \exp\left\{ \int_{t+h}^{s} F_{n+1}(\varphi_{s,r}(x), \circ \widehat{dr}) \right\} \right] \\ =&E_{\widehat{P}}\Big[v(\varphi_{s,t+h}(x), t+h) Z(t+h, s, \varphi_{s,r}(x)) \Big]. \end{aligned} \tag{3.32}$$

where $Z(t,s,x),\ s \geq t$ is given by (3.7). For notational convenience, we set

$$Q = \widehat{P}.$$

Put

$$\begin{aligned} \widehat{H}_0(s,x,\gamma,\gamma',u) &= \widehat{K}(s,x) b(s,x,\gamma,\gamma',u) + D_s \widehat{K}(s,x) \sigma(s,x,\gamma,\gamma',u) \\ &\quad + \int_{\mathbb{R}} D_{s,z} \widehat{K}(s,x) \theta(s,x,\gamma,\gamma',z,u)\, \nu(dz) \,. \end{aligned} \tag{3.33}$$

Then

$$A_1 = E\left[\int_G \int_t^T \frac{\partial}{\partial \gamma} \widehat{H}_0(s,x) \widehat{Y}(s,x)\, ds\, dx \right].$$

Differentiating with respect to h at $h=0$ we get

$$\begin{aligned} \frac{d}{dh} A_1 \Big|_{h=0} &= \frac{d}{dh} E\left[\int_G \int_t^{t+h} \frac{\partial}{\partial \gamma} \widehat{H}_0(s,x) \widehat{Y}(s,x)\, ds\, dx \right]_{h=0} \\ &\quad + \frac{d}{dh} E\left[\int_G \int_{t+h}^{T} \frac{\partial}{\partial \gamma} \widehat{H}_0(s,x) \widehat{Y}(s,x)\, ds\, dx \right]_{h=0}. \end{aligned} \tag{3.34}$$

Since $Y(t,x)=0$, we see that

$$\frac{d}{dh}E\left[\int_G\int_t^{t+h}\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0}=0. \tag{3.35}$$

Therefore by (3.32) we get

$$\begin{aligned}\frac{d}{dh}A_1\bigg|_{h=0}&=\frac{d}{dh}E\left[\int_G\int_{t+h}^T\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h,\widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t+h,s,\widehat{\varphi}_{s,r}(x)\right)\right]ds\,dx\right]_{h=0}\\&=\int_G\int_t^T\frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h,\widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t+h,s,\widehat{\varphi}_{s,r}(x)\right)\right]\right]_{h=0}ds\,dx\\&=\int_G\int_t^T\frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h,\widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right)\right]\right]_{h=0}ds\,dx.\end{aligned} \tag{3.36}$$

By (3.18)

$$\begin{aligned}v(t+h,x)=&\int_t^{t+h}L_sv(x,s)ds+\sum_{i=1}^n\int_t^{t+h}Y_i^*(x,ds)\frac{\partial v}{\partial x^i}\\&+\sum_{i=1}^n\int_t^{t+h}F_i(x,ds)\frac{\partial v}{\partial x^i}+\int_0^tF_{n+1}(x,ds)v\\&+\ \alpha\int_t^{t+h}\left\{\frac{\partial}{\partial u}b(r,x)\,dr+\frac{\partial}{\partial u}\sigma(r,x)\,dB(r)\right\}.\end{aligned} \tag{3.37}$$

Then, by (3.36) and (3.37),

$$\frac{d}{dh}A_1\bigg|_{h=0}=A_{1,1}+A_{1,2}+A_{1,3}, \tag{3.38}$$

where

$$\begin{aligned}A_{1,1}=&\int_G\int_t^T\frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right)\times\left\{\int_t^{t+h}L_s\widehat{v}(x,r)\,dr\right.\right.\right.\\&\left.\left.\left.+\int_0^tF_{n+1}(x,dr)\widehat{v}(x,r)\right\}\right]\right]_{h=0}ds\,dx,\end{aligned} \tag{3.39}$$

$$\begin{aligned}A_{1,2}=&\int_G\int_t^T\frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right)\ \times\right.\right.\\&\left.\left.\alpha\int_t^{t+h}\left\{\frac{\partial}{\partial u}b(r,\varphi_{t+h,r}(x))\,dr+\frac{\partial}{\partial u}\sigma(r,\varphi_{t+h,r}(x))dB(r)\right\}\right]\right]_{h=0}ds\,dx,\end{aligned} \tag{3.40}$$

$$\begin{aligned}A_{1,3}=&\int_G\int_t^T\frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right)\ \times\right.\right.\\&\left.\left.\left\{\sum_{i=1}^n\int_t^{t+h}Y_i^*(x,dr)\frac{\partial\widehat{v}}{\partial x^i}(x,r)+\sum_{i=1}^n\int_t^{t+h}F_i(x,dr)\frac{\partial\widehat{v}}{\partial x^i}(x,r)\right\}\right]\right]_{h=0}ds\,dx.\end{aligned} \tag{3.41}$$

Since $\widehat{Y}(t,x)=0$ we have that $v(t,x)=0$ and then

$$A_{1,1}=A_{1,3}=0.$$

By the duality formula and applying Fubini's theorem repeatedly, $A_{1,2}$ becomes

$$\begin{aligned}
A_{1,2} = & \int_G \int_t^T \frac{d}{dh} E\left[E_Q\left[\alpha \int_t^{t+h} \left\{\frac{\partial}{\partial u} b(r,\varphi_{t+h,r}(x)) I(t,s,x)\right.\right.\right. \\
& \left.\left.\left. + \frac{\partial}{\partial u}\sigma(r,\varphi_{t+h,r}(x)) D_r I(t,s,x)\right\}\right]\right]_{h=0} ds\, dx, \\
= & \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u} b(t,\varphi_{t+h,t}(x)) I(t,s,x)\right.\right.\right. \\
& \left.\left.\left. + \frac{\partial}{\partial u}\sigma(t,\varphi_{t+h,t}(x)) D_t I(t,s,x)\right\}\right]\right] ds\, dx,
\end{aligned} \tag{3.42}$$

where $I(t,s,x)=\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Z}(t,s,\widehat{\varphi}_{s,t}(x))$.
This implies that

$$\begin{aligned}
\left.\frac{d}{dh}A_1\right|_{h=0} = & A_{1,2} \\
= & \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u} b(t,\varphi_{t,t}(x)) I(t,s,x)\right.\right.\right. \\
& \left.\left.\left. + \frac{\partial}{\partial u}\sigma(t,\varphi_{t,t}(x)) D_t I(t,s,x)\right\}\right]\right] ds\, dx. \\
= & \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u} b(t,x) I(t,s,x) + \frac{\partial}{\partial u}\sigma(t,x) D_t I(t,s,x)\right\}\right]\right] ds\, dx,
\end{aligned} \tag{3.43}$$

where the last equality follows from the fact that $\widehat{\varphi}_{t,t}(x)=x$. Moreover, we see that

$$\left.\frac{d}{dh}A_2\right|_{h=0} = \int_G E\left[\alpha\left\{\frac{\partial}{\partial u} b(t,x)\widehat{K}(t,x) + \frac{\partial}{\partial u}\sigma(t,x) D_t\widehat{K}(t,x) + \frac{\partial}{\partial u} f(t,x)\right\}\right] ds\, dx. \tag{3.44}$$

Then, using the adjoint operators L^* and ∇_x^* (see (3.8)) we get

$$\begin{aligned}
A_3 &= E\left[\int_G \int_t^T \widehat{K}(s,x) L\widehat{Y}^{\beta_\alpha}(s,x) dx\, dt\right] \\
&= E\left[\int_G \int_t^T L^*\widehat{K}(s,x)\widehat{Y}^{\beta_\alpha}(s,x)\, dx\, dt\right], \\
A_4 &= E\left[\int_G \int_t^T \left\{\widehat{K}(s,x)\nabla_{\gamma'} b(s,x) + D_s\widehat{K}(s,x)\nabla_{\gamma'}\sigma(s,x)\right.\right. \\
&\quad \left.\left. + \int_{\mathbb{R}} D_{s,z}\widehat{K}(s,x)\nabla_{\gamma'}\theta(s,x,z)\nu(dz)\right\}\nabla_x\widehat{Y}^{\beta_\alpha}(s,x)\, ds\, dx\right] \\
&= E\left[\int_G \int_t^T \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}^{\beta_\alpha}(s,x)\, ds\, dx\right].
\end{aligned}$$

Differentiating with respect to h at $h=0$ gives

$$\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G\int_t^{t+h} L^*\widehat{K}(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} \\
&\quad +\frac{d}{dh}E\left[\int_G\int_{t+h}^T L^*\widehat{K}(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0}, \qquad (3.45)\\
\frac{d}{dh}A_4\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G\int_t^{t+h} \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} \\
&\quad +\frac{d}{dh}E\left[\int_G\int_{t+h}^T \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0}. \qquad (3.46)
\end{aligned}$$

Using the same arguments as before, it can be shown that

$$\frac{d}{dh}A_3\Big|_{h=0} = \int_G\int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)I_1(t,s,x)+\frac{\partial}{\partial u}\sigma(t,x)D_tI_1(t,s,x)\right\}\right]\right]ds\,dx, \tag{3.47}$$

$$\frac{d}{dh}A_4\Big|_{h=0} = \int_G\int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)I_2(t,s,x)+\frac{\partial}{\partial u}\sigma(t,x)D_tI_2(t,s,x)\right\}\right]\right]ds\,dx, \tag{3.48}$$

where $I_1(t,s,x)=L^*\widehat{K}(s,x)\widehat{Z}(t,s,\varphi_{s,t}(x))$ and $I_2(t,s,x)=\nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Z}(t,s,\varphi_{s,t}(x))$. Therefore, differentiating (3.29) with respect to h at $h=0$ yields

$$\begin{aligned}
&E_Q\left[E\left[\alpha\int_G\left\{\frac{\partial}{\partial u}f(t,x)+\left(\widehat{K}(t,x)+\int_t^T\Big(I(t,s,x)+I_1(t,s,x)+I_2(t,s,x)\Big)ds\right)\frac{\partial}{\partial u}b(t,x)\right.\right.\right.\\
&\qquad\left.\left.\left.+D_t\left(\widehat{K}(t,x)+\int_t^T\Big(I(t,s,x)+I_1(t,s,x)+I_2(t,s,x)\Big)ds\right)\frac{\partial}{\partial u}\sigma(t,x)\right\}dx\right]\right]=0.
\end{aligned}\tag{3.49}$$

By the definition of $\widehat{p}(t,x)$, we have

$$\widehat{p}(t,x)=\widehat{K}(t,x)+\int_t^T\Big(I(t,s,x)+I_1(t,s,x)+I_2(t,s,x)\Big)ds.$$

We can then write (3.49), as

$$\begin{aligned}
&E_Q\left[E\left[\int_G\frac{\partial}{\partial u}\left\{f(t,x,\Gamma,\widehat{u},\omega)+p(t,x)b(t,x,\Gamma,\Gamma',\widehat{u},\omega)+D_tp(t,x)\sigma(t,x,\Gamma,\Gamma',\widehat{u},\omega)\right.\right.\right.\\
&\left.\left.\left.+\int_{\mathbb{R}}D_{t,z}p(t,x)\theta(t,x,\Gamma,\Gamma',\widehat{u},z,\omega)\nu(dz)\right\}\alpha\,dx\right]\right]\\
=\;& 0.
\end{aligned}$$

Since this holds for all bounded $\mathcal{E}_t-$measurable random variables α, we conclude that

$$E_Q\left[E\left[\int_G\frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))\,dx\,\Big|\,\mathcal{E}_t\right]\right]=0 \text{ a.e. in } (t,x,\omega),$$

means

$$E\left[E_Q\left[\int_G \frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))\,dx\right]\middle|\,\mathcal{E}_t\right]=0 \text{ a.e. in } (t,x,\omega),$$

which completes the proof. ■

4 Applications

In this Section we take aim at two applications of Theorem 4 : The first one pertains to partial information optimal harvesting, whereas the other one refers to portfolio optimization under partial observation.

4.1 Partial information optimal harvesting

Assume that $\Gamma(t,x)$ describes the density of a population (e.g. fish) at time $t\in(0,T)$ and at the location $x\in G\subset\mathbb{R}^d$. Further suppose that $\Gamma(t,x)$ is modeled by the stochastic-reaction diffusion equation

$$\begin{aligned} d\Gamma(t,x) &= \left[\frac{1}{2}\Delta\Gamma(t,x)+b(t)\Gamma(t,x)-c(t)\right]dt+\sigma(t)\Gamma(t,x)\,dB(t) \\ &\quad+\int_{\mathbb{R}}\theta(t,z)\Gamma(t,x)\widetilde{N}(dz,dt),\ (t,x)\in[0,T]\times G, \qquad (4.1)\\ &\quad \text{where } \Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial X_i^2} \text{is the Laplacian,} \end{aligned}$$

with boundary condition

$$\begin{aligned} \Gamma(0,x) &= \xi(x),\ x\in\overline{G}, \\ \Gamma(t,x) &= \eta(t,x),\ (t,x)\in(0,T)\times\partial G. \end{aligned}$$

where b, σ, θ, c are given processes such that D1–D10 in Section 3.2 are fulfilled.

The process $c(t)\geq 0$ is our harvesting rate, which is assumed to be a $\mathcal{E}_t-$predictable admissible control.

We aim to maximize both expected cumulative utility of consumption and the terminal size of the population subject to the performance functional

$$J(c)=E\left[\int_G\int_0^T\zeta(s)U(c(s))\,ds\,dx+\int_G\xi\Gamma^{(c)}(T,x)\,dx\right], \qquad (4.2)$$

where $U:[0,+\infty)\longrightarrow\mathbb{R}$ is a C^1 utility function, $\zeta(s)=\zeta(s,x,\omega)$ is an $\mathcal{F}_t-$predictable process and $\xi=\xi(\omega)$ is an $\mathcal{F}_T-$measurable random variable such that

$$E\left[\int_G|\zeta(t,x)|\,dx\right]<\infty \text{ and } E\left[\xi^2\right]<\infty.$$

We want to find an admissible control $\hat{c} \in A_{\mathcal{E}}$ such that

$$\sup_{c \in A_{\mathcal{E}}} J(c) = J(\hat{c}). \tag{4.3}$$

Note that condition 1 of Section 3.2 is fulfilled. Using the same arguments in [2] it can be verified that the linear SPDE (4.1) also satisfies conditions 2–4. Using the previous notation, we note that in this case, with $u = c$,

$$f(t, x, \Gamma(t, x), c(t), \omega) = \zeta(s, \omega)U(c(t)); \;\; g(x, \Gamma(t, x), \omega) = \xi(\omega)\Gamma^{(c)}(t, x).$$

Hence

$$\begin{aligned}
K(t, x) =& \frac{\partial}{\partial\gamma} g(x, \Gamma(T, x), \omega) + \int_t^T \frac{\partial}{\partial\gamma} f(s, x, \Gamma(s, x), u(s, x), \omega)\, ds = \xi(\omega),\\
H_0(t, x, \gamma, c) =& \xi(\omega)\left(b(t, x)\gamma - c\right) + D_t\xi(\omega)\sigma(t)\gamma + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t, z)\gamma\, \nu(dz)\, dt,\\
I(t, s, x) =& \left(b(t, x)\xi(\omega) + D_t\xi(\omega)\sigma(t) + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t, z)\, \nu(dz)\right) \times Z(t, s, \widehat{\varphi}_{s,t}(x)),\\
I_1(t, s, x) =& I_2(t, s, x) = 0,\\
Z(s, t, x) =& \exp\left\{\int_t^s F_{n+1}(x, \circ\widehat{dr})\right\},\\
F_{n+1}(x, dt) =& b(t)\, dt + \sigma(t)\, dB(t) + \int_{\mathbb{R}_0} \theta(t, z)\widetilde{N}(dt, dz).
\end{aligned}$$

In this case we have $\varphi_{s,t}(x) = x$ since $K(s, x) = \xi(\omega)$ if follows that $L^*K(s, x) = 0$, in addition, H_0 does not depend on γ' and then $\nabla_x^*\left(\nabla_{\gamma'}H_0(s, x, \Gamma(s, x), \nabla_x\Gamma(s, x), u(s, x))\right) = 0$. Therefore

$$p(t, x) = \xi(\omega) + \int_t^T \left(b(t, x)\xi(\omega) + D_t\xi(\omega)\sigma(t) + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t, z)\nu(dz)\right) Z(t, r, \widehat{\varphi}_{s,t}(x))\, dr, \tag{4.4}$$

and the Hamiltonian becomes

$$\begin{aligned}
H(t, x, \gamma, c) \;=\;& \zeta(t)U(c) + p(t, x)\left(b(t, x)\Gamma(t, x) - c(t)\right) + D_t p(t, x)\sigma(t)\\
&+ \int_{\mathbb{R}_0} D_{t,z} p(t, x)\theta(t, z)\nu(dz).
\end{aligned} \tag{4.5}$$

Then, $\hat{c} \in \mathcal{A}_{\mathcal{E}}$ is an optimal control for the problem (4.3) if we have:

$$\begin{aligned}
0 \;&=\; E\left[E_Q\left[\int_G \frac{\partial}{\partial c} H(t, x, \hat{\Gamma}(t, x), \hat{c}(t))\, dx\right]\middle|\, \mathcal{E}_t\right]\\
&=\; E\left[E_Q\left[\int_G \left\{\zeta(t)U'(\hat{c}(t)) - p(t, x)\right\} dx\right]\middle|\, \mathcal{E}_t\right]\\
&=\; U'(\hat{c}(t))E\left[E_Q\left[\int_G \zeta(t, x)\, dx\right]\middle|\, \mathcal{E}_t\right] - E\left[E_Q\left[\int_G p(t, x)\, dx\right]\middle|\, \mathcal{E}_t\right].
\end{aligned}$$

We have proved a theorem similar to Theorem 4.2 in [14]:

Theorem 6 *If there exists an optimal harvesting rate $\hat{c}(t)$ of problem (4.3), then it satisfies the equation*

$$U'(\hat{c}(t))E\left[E_Q\left[\int_G \zeta(t,x)\,dx\right]\middle|\,\mathcal{E}_t\right] = E\left[E_Q\left[\int_G p(t,x)\,dx\right]\middle|\,\mathcal{E}_t\right]. \tag{4.6}$$

4.2 Application to optimal stochastic control of jump diffusion with partial observation

In this Subsection we want to apply ideas of non-linear filtering theory in connection with Theorem 4 to solve a portfolio optimization problem, where the trader has limited access to market information (Section 4.3). As for general background information on non-linear filtering theory the reader may e.g. consult [2]. For the concrete setting that follows below see also [13] and [15].

Suppose that the state process $X(t) = X^{(u)}(t)$ and the observation process $Z(t)$ are described by the following system of SDE's:

$$\begin{aligned} dX(t) =& \alpha(X(t),u(t))\,dt + \beta(X(t),u(t))\,dB^X(t), \\ dZ(t) =& h(t,X(t))\,dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi\, N_\lambda(dt,d\xi), \end{aligned} \tag{4.7}$$

where $(B^X(t); B^Z(t)) \in \mathbb{R}^2$ is a Wiener process independent of the initial value $X(0)$, and N_λ is an integer valued random measure with predictable compensator

$$\mu(dt,d\xi,\omega) = \lambda(t,X_t,\xi)\,dt\,\nu(d\xi),$$

for a Lévy measure ν and an intensity rate function $\lambda(t,x,\xi)$, such that the increments of N_λ are conditionally independent with respect to the filtration generated by B_t^X. Further $u(t)$ is a control process which takes values in a closed convex set $U \subset \mathbb{R}$ and which is adapted to the filtration $\mathcal{G}_t$ generated by the observation process $Z(t)$. The coefficients $\alpha : \mathbb{R}\times U \longrightarrow \mathbb{R}$, $\beta : \mathbb{R}\times U \longrightarrow \mathbb{R}$, $\lambda : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0 \longrightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are twice continuously differentiable.

In what follows we shall assume that a strong solution $X_t = X_t^{(u)}$ of (4.7), if it exists, takes values in a given Borel set $G \subseteq \mathbb{R}$. Let us introduce the performance functional

$$J(u) := E\left[\int_0^T f(X(t),Z(t),u(t))\,dt + g(X(T),Z(T))\right],$$

where $f : G\times\mathbb{R}\times U \longrightarrow \mathbb{R}$, $g : G\times\mathbb{R} \longrightarrow \mathbb{R}$ are (lower) bounded C^1 functions. We want to find the maximizer u^* of J, that is

$$J^* = \sup_{u\in\mathcal{A}} J(u) = J(u^*), \tag{4.8}$$

where $\mathcal{A}$ is the set of admissible controls consisting of $\mathcal{G}_t$−predictable controls u such that (4.7) admits a unique strong solution.

We shall now briefly outline how the optimal control problem (4.8) for SDE's with partial observation can be transformed into one for SPDE's with complete information. See e.g. [2] and [13] for details. In the sequel we assume that $\lambda(t, x, \xi) > 0$ for all t, x, ξ and that the exponential process

$$\begin{aligned} M_t \quad := \quad & \exp\left\{\int_0^t h(X(s))\, dB^Z(s) - \frac{1}{2}\int_0^t h^2(X(s))\, ds \right. \\ & \left. + \int_0^t \int_{\mathbb{R}_0} \log \lambda(s, X(s), \xi) N_\lambda(ds, d\xi) + \int_0^t \int_{\mathbb{R}_0} [1 - \lambda(s, X(s), \xi)]\, ds\, \nu(d\xi)\right\}; \quad t \geq 0 \end{aligned}$$

is well defined and a martingale. Define the change of measure

$$dQ' = M_T dP$$

and set

$$N_T = M_T^{-1}$$

Using the Girsanov theorem for random measures and the uniqueness of semimartingale characteristics (see, e.g. [10]), one sees that the processes (4.7) get decoupled under the measure Q' in the sense that system (4.7) transforms to

$$\begin{aligned} dX(t) &= \alpha(X(t), u(t))\, dt + \beta(X(t), u(t))\, dB^X(t), \\ dZ(t) &= dB(t) + dL(t), \end{aligned}$$

where $Z(t)$ is a Lévy process independent of Brownian motion $B^X(t)$, and consequently independent of $X(t)$, under Q'. Here

$$B(t) = B^Z(t) - \int_0^t h(X(s))\, ds$$

is the Brownian motion part and

$$L(t) = \int_0^t \int_{\mathbb{R}_0} \xi N(dt, d\xi)$$

is the pure jump component associated to the Poisson random measure $N(dt, d\xi) = N_\lambda(dt, d\xi)$ with compensator given by $ds\nu(d\xi)$. Define the differential operator $A = A_{z,u}$ by

$$A\phi(x) = A_u\phi(x) = \alpha(x, u)\frac{d\phi}{dx}(x) + \frac{1}{2}\beta^2(x, u)\frac{d^2\phi}{dx^2}(x)$$

for $\phi \in C_0^2(\mathbb{R})$. Hence A_u is the generator of $X(t)$, if u is constant. Set

$$a(x, u) = \frac{1}{2}\beta^2(x, u). \tag{4.9}$$

Then the adjoint operator A^* of A is given by

$$A^*\phi = \frac{\partial}{\partial x}\left(a(x, u)\frac{d\phi}{dx}(x)\right) + \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x}(x, u)\phi(x)\right) - \frac{\partial}{\partial x}\left(a(x, u)\phi(x)\right). \tag{4.10}$$

Let us assume that the initial condition $X(0)$ has a density p_0 and that there exists a unique strong solution $\Phi(t,x)$ of the following SPDE (Zakai equation)

$$d\Phi(t,x) = A^*\Phi(t,x)\,dt + h(x)\Phi(t,x)\,dB(t) + \int_{\mathbb{R}_0} [\lambda(t,x,\xi) - 1]\Phi(t,x)\,\widetilde{N}(dt,d\xi), \tag{4.11}$$

with

$$\Phi(0,x) = p_0(x).$$

Then $\Phi(t,x)$ is the *unnormalized conditional density* of $X(t)$ given $\mathcal{G}_t$ and satisfies:

$$E_{Q'}[\phi(X(t))N_t|\mathcal{G}_t] = \int_{\mathbb{R}} \phi(x)\Phi(t,x)dx \tag{4.12}$$

for all $\phi \in C_b(\mathbb{R})$.
Using (4.12) and (4.11) under the change of measure Q' and the definition of the performance functional we obtain that

$$\begin{aligned}
J(u) &= E\left[\int_0^T f(X(t),Z(t),u(t))dt + g(X(T),Z(T))\right] \\
&= E_{Q'}\left[\left\{\int_0^T f(X(t),Z(t),u(t))dt + g(X(T),Z(T))\right\} N_T\right] \\
&= E_{Q'}\left[\int_0^T f(X(t),Z(t),u(t))N_t dt + g(X(T),Z(T))N_T\right] \\
&= E_{Q'}\left[\int_0^T E_Q\left[f(X(t),Z(t),u(t))N_t \mid \mathcal{G}_t\right] dt + E_Q\left[g(X(T),Z(T))N_T \mid \mathcal{G}_t\right]\right] \\
&= E_{Q'}\left[\int_0^T \int_G f(x,Z(t),u(t))\Phi(t,x)dxdt + \int_G g(x,Z(T))\Phi(T,x)dx\right].
\end{aligned}$$

The observation process $Z(t)$ is a Q'-Lévy process. Hence the partial observation control problem (4.8) reduces to a SPDE control problem under complete information. More precisely, our control problem is equivalent to the maximization problem

$$\sup_u E_{Q'}\left[\int_0^T \int_G f(x,Z(t),u(t))\Phi(t,x)\,dx\,dt + \int_G g(x,Z(T))\Phi(T,x)\,dx\right], \tag{4.13}$$

where Φ solves the SPDE (4.11). So the latter problem can be tackled by means of the maximum principle of Section 2.
For convenience, let us impose that a in (4.9) is independent of the control, i.e.

$$a(x,u) = a(x)\,.$$

Denote by $\mathcal{A}_1$ the set $u \in \mathcal{A}$ for which (4.11) has a unique solution. Consider the general stochastic Hamiltonian (if existent) of the control problem (4.13) given by

$$\begin{aligned}
H(t,x,\phi,\phi',u,\omega) = {} & f(t,x,Z(t),u)\phi + p(t,x)b(t,x,\phi,\phi',u) + D_t p(t,x)h(x)\phi \\
& + \int_{\mathbb{R}_0} D_{t,z}p(t,x)[\lambda(t,x,\xi) - 1]\phi\,\nu(dz),
\end{aligned} \tag{4.14}$$

where

$$b(t,x,\phi,\phi',u) = \left(\frac{d^2a}{dx^2}(x) - \alpha(x,u)\right)\phi + \left(\frac{da}{dx}(x) - \alpha(x,u)\right)\phi'$$

and where $p(t,x)$ is defined as in (3.22) with

$$g(x,\phi,\omega) = g(x,Z(T))\phi$$

and

$$L\psi(x) = a(x)\frac{d^2\psi}{dx^2}(x),\ \ \psi \in C_0^2(\mathbb{R}).$$

Assume that the conditions (1)-(5) in Section 3.2 are satisfied with respect to (4.13) for controls $u \in \mathcal{A}_1$. Then by the general stochastic maximum principle (Theorem 4) applied to the partial information control problem (4.8) we find that

$$E\left[E_Q\left[\int_G \frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Phi},\widehat{\Phi}',\hat{u},\omega)\,dx\,\middle|\,\mathcal{G}_t\right]\right] = 0, \tag{4.15}$$

if $\hat{u} \in \mathcal{A}_1$ is an optimal control.

4.3 Optimal consumption with partial observation

Let us illustrate the maximum principle by inquiring into the following portfolio optimization problem with partial observation: Assume the wealth $X(t)$ at time t of an investor is modeled by

$$dX(t) = [\mu X(t) - u(t)]\,dt + \sigma X(t)\,dB^X(t), 0 \le t \le T,$$

where $m \in \mathbb{R}, \sigma \neq 0$ are constants, $B^X(t)$ a Brownian motion and $u(t) \ge 0$ the *consumption rate*. Suppose that the initial value $X(0)$ has the density $p_0(x)$ and that $u(t)$ is adapted to the filtration $\mathcal{G}_t$ generated by the observation process

$$dZ(t) = mX(t)\,dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi\, N_\lambda(dt,d\xi),\ \ Z(0) = 0\,,$$

where m is a constant. As before we require that $(B^X(t), B^Z(t))$ is a Brownian motion independent of the initial value $X(0)$, and that N_λ is an integer valued random measure as described in (4.7). Further, let us restrict the wealth process $X(t)$ to be bounded from below by a threshold $\zeta > 0$ for $0 \le t \le T$. The investor intends to maximize the expected utility of his consumption and terminal wealth according to the performance criterion

$$J(u) = E\left[\int_0^T \frac{u^r(t)}{r}\,dt + \theta X^r(T)\right]\,,\ r \in (0,1),\ \ \theta > 0. \tag{4.16}$$

So we are dealing with a partial observation control problem of the type (4.8) (for $G = [\zeta,\infty)$). Here, the operator A in (4.11) has the form

$$A\phi(x) = \frac{1}{2}\sigma^2x^2\phi''(x) + [\mu x - u]\phi'(x), \tag{4.17}$$

(where $'$ denotes the differentiation which respect to x) and hence

$$A^*\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) - [\mu x - u]\phi'(x) - \mu\phi(x)\,. \tag{4.18}$$

Therefore the Zakai equation becomes

$$\begin{aligned} d\Phi(t,x) =& \left[\frac{1}{2}\sigma^2 x^2 \Phi''(t,x) - [\mu x - u]\,\Phi'(t,x) - \mu\Phi(t,x)\right] dt + x\Phi(t,x)\,dB(t) \\ &+ \int_{\mathbb{R}_0} [\lambda(t,x,\xi) - 1]\Phi(t,x)\,\widetilde{N}(dt,d\xi), \\ \Phi(0,x) =& \,p_0(x)\,,\quad x > \zeta, \\ \Phi(t,0) =& \,0\,,\quad t \in (0,T), \end{aligned} \tag{4.19}$$

where $\widetilde{N}(dt,d\xi)$ is a compensated Poisson random measure under the corresponding measure Q'. Since $L\psi = \frac{1}{2}\sigma^2 x^2 \frac{d^2\psi}{dx^2}(x)$ is uniformly elliptic for $x > \zeta$ there exists a unique strong solution of (4.19). Further one verifies that condition (4) of Section 3.2 is fulfilled. See [2]. So our problem amounts to finding an admissible $\hat{u} \in \mathcal{A}_1$ such that

$$J_1(\hat{u}) = \sup_{u \in \mathcal{A}_1} J_1(u), \tag{4.20}$$

where

$$J_1(u) = E_{Q'}\left[\int_0^T \int_G \frac{u^r(t)}{r}\Phi(t,x)\,dx\,dt + \int_V \theta x^r \Phi(T,x)\,dx\right].$$

Our assumptions imply that condition (1) of Section 3.2 holds. Further, by exploiting the linearity of the SPDE (4.19) one shows as in [3] that also the conditions (2)-(4) in Section

3.2 are fulfilled. Using the notation of (4.14) we see that

$$
\begin{aligned}
f(x, Z(t), u(t)) =& \frac{u^r(t)}{r}, \\
g(x, Z(T)) =& \theta x^r, \\
L^*\Phi(t,x) =& \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\Phi(t,x), \\
K(t,x) =& \theta x^r(T) + \int_t^T \frac{u^r(s)}{r} ds, \\
H_0(t,x,\phi,\phi',u) =& \left[(-\mu x - u)\phi'(t,x) - \mu\phi(t,x)\right] K(t,x) + D_t K(t,x) x\phi \\
& + \int_{\mathbb{R}_0} D_{t,z}K(t,x)[\lambda(t,x,\xi)-1]\phi\,\nu(d\xi) \\
I(t,s,x) =& \left(-\mu K(s,x) + D_s K(s,x) x + \int_{\mathbb{R}_0} D_{s,z}K_1(s,x)[\lambda(s,x,\xi)-1]\nu(d\xi)\right) \times \\
& Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
I_1(t,s,x) =& \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} K(s,x) \times Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
I_2(t,s,x) =& \frac{\partial}{\partial x}\left[(-\mu x - u)K(s,x)\right] Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
Z(t,s,x) =& \exp\left\{\int_t^s F_{n+1}\left(x, \circ\widehat{dr}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
F_{n+1}(x,dt) =& \mu dt + x dB(t) + \int_{\mathbb{R}_0}\left[\lambda(t,x,\xi)-1\right]\widetilde{N}(dt,d\xi), \\
F_i(x,dt) = F(x,dt) =& -\left[\mu x - u\right] dt, \quad i = 1,\cdots,n.
\end{aligned}
$$

In this case we have $\varphi_{s,t}(x) = x + \int_t^s G(\varphi_{s,r}(x), \circ dr)$, where $G(x,t) = X(x,t) + F(t,x)$. Then

$$
p(t,x) = K(t,x) + \int_t^T \Big(I_1(r,s,x) + I_2(r,s,x) + I_3(r,s,x)\Big) dr. \tag{4.21}
$$

So the Hamiltonian (if it exists) becomes

$$
\begin{aligned}
H(t,x,\phi,\phi',u) \quad = \quad & \frac{u^r(t)}{r}\phi + \left[(-\mu x - u)\phi'(t,x) - \mu\phi(t,x)\right] p(t,x) \\
& + D_t p(t,x) x\phi + \int_{\mathbb{R}_0} D_{t,z}p(t,x)[\lambda(t,x,\xi)-1]\phi\,\nu(d\xi).
\end{aligned}
$$

Hence, if $\hat{u}$ is an optimal control of the problem (4.8) such that the Hamiltonian is well-defined, then it follows from (4.15) and (4.12) that

$$
\begin{aligned}
0 \quad = \quad & E_{Q'}\left[E_Q\left[\int_G \frac{\partial}{\partial u} H(t,x,\widehat{\Phi},\widehat{\Phi}',\widehat{u})\,dx\right]\Bigg|\,\mathcal{G}_t\right] \\
= \quad & E_{Q'}\left[E_Q\left[\int_G \left\{u^{r-1}(t)\widehat{\Phi}(t,x) + \widehat{\Phi}'(t,x)\widehat{p}(t,x)\right\} dx\right]\Bigg|\,\mathcal{G}_t\right].
\end{aligned}
$$

Thus we get

$$u^{r-1}(t) = -\frac{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}'(t,x)\widehat{p}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}(t,x)dx\right]\Big|\,\mathcal{G}_t\right]}.$$

Using integration by parts and (4.12) implies that

$$\begin{aligned}
u^*(t) &= \left(-\frac{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}'(t,x)\widehat{p}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}\right)^{\frac{1}{r-1}} \\
&= \left(\frac{E_Q\left[E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\widehat{p}'(t,x)\,dx\Big|\,\mathcal{G}_t\right]\right]}{E_Q\left[E_{Q'}\left[\int_G \widehat{\Phi}(t,x)dx\Big|\,\mathcal{G}_t\right]\right]}\right)^{\frac{1}{r-1}} \\
&= \left(E_Q\left[\frac{E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\widehat{p}'(t,x)\,dx\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\,dx\Big|\,\mathcal{G}_t\right]}\right]\right)^{\frac{1}{r-1}} \\
&= \left(E_Q\left[\frac{E_Q\left[\widehat{p}'(t,X(t))N_t|\,\mathcal{G}_t\right]}{E_Q\left[N_t|\,\mathcal{G}_t\right]}\right]\right)^{\frac{1}{r-1}} \\
&= E_Q\left[E\left[\widehat{p}'(t,X(t))\big|\,\mathcal{G}_t\right]\right]^{\frac{1}{r-1}}.
\end{aligned}$$

So if $u^*(t)$ maximizes (4.16) then $u^*(t)$ necessarily satisfies

$$\begin{aligned}
u^*(t) &= E_Q\left[E\left[\widehat{p}'(t,X(t))\big|\,\mathcal{G}_t\right]\right]^{\frac{1}{r-1}} \\
&= E\left[E_Q\left[\widehat{p}'(t,X(t))\right]\big|\,\mathcal{G}_t\right]^{\frac{1}{r-1}}.
\end{aligned} \tag{4.22}$$

Theorem 7 *Suppose that $\widehat{u} \in \mathcal{A}_{\mathcal{G}_t}$ is an optimal portfolio for the partial observation control problem*

$$\sup_{u\in\mathcal{A}_{\mathcal{G}_t}} E\left[\int_0^T \frac{u^r(t)}{r}dt + \theta X^r(T)\right],\ r\in(0,1),\ \ \theta>0,$$

with the wealth and the observation processes $X(t)$ and $Z(t)$ at time t given by

$$\begin{aligned}
dX(t) &= [\mu X(t) - u(t)]\,dt + \sigma X(t)dB^X(t), 0\le t\le T, \\
dZ(t) &= mX(t)dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi N_\lambda(dt,d\xi).
\end{aligned}$$

Then

$$u^*(t) = E\left[E_Q\left[\widehat{p}'(t,X(t))\right]\big|\,\mathcal{G}_t\right]^{\frac{1}{r-1}}. \tag{4.23}$$

Remark 8 *Note that the last problem cannot be treated within the framework of [14], since the random measure $N_\lambda(dt,d\xi)$ is not necessarily a functional of a Lévy process. Let us also mention that the SPDE maximum principle studied in [17] does not apply to our optimal consumption with partial observation problem. This is due to the fact the corresponding Hamiltonian in [17] fails to be concave.*

References

[1] Baghery, F., Øksendal, B.: A maximum principle for stochastic control with partial information. Stochastic Analysis and Applications 25, p. 705-717 (2007).

[2] Bensousssan, A.: Maximum principle and dynamic programming Approaches of the optimal control of partially observed diffusions. Stochastics 9 (1983).

[3] Bensoussan, A: Stochastic Control of Partially Observable Systems. Cambridge University Press (1992).

[4] Carmona, R., Nualart, D.: Nonlinear Stochastic Integrators, Equations and Flows. Stochastics Monographs, Vol. 6, Gordon and Breach Science Publishers (1990).

[5] Carmona, R., Tehranchi, M.: Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective. Springer (2006).

[6] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press (1992).

[7] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy processes with Applications to Finance. Universitext Springer, (2008).

[8] Di Nunno, G., Meyer-Brandis, T., Øksendal, B., Proske, F.: Malliavin calculus and anticipative Itô formulae for Lévy processes. Inf. dim. Anal. Quant. Probab. 8, p. 235-258 (2005).

[9] Framstad, N., Øksendal, B., Sulem, A.: Stochastic maximum principle for optimal control of jump diffusions and applications to finance. J. Optimization Theory and Appl. 121 (1), p. 77-98 (2004). Errata: J. Optimization Theory and Appl. 124 (2), p. 511-512 (2005).

[10] Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Springer (1987).

[11] Kallianpur, G., Xiong, J.: Stochastic Differential Equations in Infinite Dimensional Spaces. IMS Lecture Notes-Monograph Series, Vol. 26 (1995).

[12] Kunita, H.: Stochastic flows and Stochastic differential equations. Cambridge University Press (1990).

[13] Mandrekar, V., Meyer-Brandis T., Proske F.: A Bayes formula for non-linear filtering with Gaussian and Cox noise, Pure Mathematics Preprint series, University of Oslo (2007).

[14] Meyer-Brandis, T., Øksendal, B., Zhou, X.: A Malliavin calculus approach to a general maximum principle for stochastic control. Submitted and accepted in Stochastics. Eprint, Pure Mathematics Preprint series, University of Oslo, 10/2008.

[15] Meyer-Brandis, T., Proske, F.: Explicit solution of a non-linear filtering problem for Lévy processes with application to finance. Appl. Math. Optim. 50, p. 119-134 (2004).

[16] Nualart, D.: The Malliavin Calculus and Related Topics. Second edition. Springer (2006).

[17] Øksendal, B., Proske, F., Zhang, T.: Backward stochastic partial differential equations with jumps and application to optimal control of random jump fields. Stochastics 77 (5), p. 381-399 (2005).

[18] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: The Mathematical Theory of Optimal Processes. John Wiley & Sons, Inc., New York, 1962, viii + 360 pp.

[19] Sanz-Solé, M.: Malliavin Calculus with Applications to Stochastic Partial Differential Equations. CRC Press (2005).

[20] Tang, S.: The maximum principle for partially observed optimal control of stochastic differential equations.. SIAM Journal on Control and Optimization 36 (5), p. 1596-1617 (1998).

[21] Zhou, X.: On the necessary conditions of optimal controls for stochastic partial differential equations. SIAM Journal on Control and Optimization 31, 86), p. 1462-1478 (1993).