The Fatou Closedness under Model Uncertainty

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Abstract

We provide a characterization in terms of Fatou closedness for weakly closed monotone convex sets in the space of $\mathcal{P}$-quasisure bounded random variables, where $\mathcal{P}$ is a (possibly non-dominated) class of probability measures. We illustrate the relevance of our results by applications in the field of mathematical finance.

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1 Introduction

A fundamental result attributed to Grothendieck ([Gr54, p321, Exercise 1]) and based on the Krein-Smulian theorem (see [DS58, Theorem V.5.7]) characterizes weak*-closedness of a convex subset of $L^\infty_P := L^\infty(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space, by means of a property called Fatou closedness as follows:

Theorem 1.1. *Let $\mathcal{A} \subset L^\infty_P$ be convex. Equivalent are:*

(i) $\mathcal{A}$ is weak-closed (i.e. closed in $\sigma(L^\infty_P, L^1_P)$).*

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(ii) $\mathcal{A}$ *is Fatou closed, i.e. if* $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *is a bounded sequence which converges* P*-almost surely to* X*, then* $X \in \mathcal{A}$*.*

Theorem 1.1 is very useful and often applied in the mathematical finance literature such as in the classic proof of the Fundamental Theorem of Asset Pricing – see e.g. [DS94] or [DS06] – or in the dual representation of convex risk functions – see e.g. [FS04]. In all cases the problem is that the norm dual of L^∞_P contains undesired singular elements, whereas in the weak*-duality $(L^\infty_P, \sigma(L^\infty_P, L^1_P))$ the elements of the dual space are identified with σ-additive measures. However, as the weak*-topology is not first-countable, verifying that some set is weak*-closed is in general quite challenging. This is where Theorem 1.1 proves helpful.

The aim of this paper is to study the existence of a version of Theorem 1.1 for the case when the probability measure P is replaced by a class $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$. In general this class $\mathcal{P}$ does not allow for a dominating probability. Applications of such a result are for instance robust versions of the Fundamental Theorem of Asset Pricing and the Superhedging Duality Theorem as well as dual representations of convex risk functions in robust frameworks as studied in [BK12, BN15, BFM15, Nu14]. These kind of frameworks have become increasingly popular in the mathematical finance literature to describe a decision maker who has to deal with the uncertainty which arises from model ambiguity. Here the class of probability models $\mathcal{P}$ the decision maker takes into account represents her degree of ambiguity about the right probabilistic model. If $\mathcal{P} = \{P\}$ there is no ambiguity. In many models which account for model ambiguity $\mathcal{P}$ in fact turns out to be a non-dominated class of probability measures.

We will show that there is a version of Theorem 1.1 in a robust probabilistic framework $(\Omega, \mathcal{F}, \mathcal{P})$, see Theorem 3.9. Let

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{F},$$

denote the capacity generated by $\mathcal{P}$. Under some conditions on the convex set $\mathcal{A}$ and on L^∞_c we obtain equivalence between

(WC) $\mathcal{A} \subset L^\infty_c$ is $\sigma(L^\infty_c, ca_c)$-closed,

(FC) $\mathcal{A} \subset L^\infty_c$ is Fatou closed: for any bounded sequence $\{X_n\} \subset \mathcal{A}$ and $X \in L^\infty_c$ such that $X_n \to X$ $\mathcal{P}$-quasi surely we have that $X \in \mathcal{A}$,

where L^∞_c and ca_c are the robust analogues of L^∞_P and L^1_P given by the capacity c, respectively, and $\mathcal{P}$ quasi sure convergence means Q-almost sure convergence under each $Q \in \mathcal{P}$. The conditions we have to require on $\mathcal{A}$ are monotonicity and a property called $\mathcal{P}$-sensitivity. Monotonicity is typically satisfied, at least in economic applications, and we show that $\mathcal{P}$-sensitivity is indeed a necessary condition to have (WC) $\Leftrightarrow$ (FC). If $\mathcal{P}$ is dominated, $\mathcal{P}$-sensitivity is always fulfilled. Another issue is that L_c^∞ is in general not order complete, which means that a bounded family of random variables in L_c^∞ does not admit an essential supremum. However, this property will be crucial for proving (WC) $\Leftrightarrow$ (FC), and it corresponds to aggregation type results as in [Co12, STZ11]. Asking for the existence of an essential supremum in L_c^∞, as we will have to, is equivalent to requiring that the dual space of ca_c can be identified with L_c^∞, see Proposition 3.10.

We also provide a counter example showing that for non-dominated $\mathcal{P}$ there is no proof of (WC) $\Leftrightarrow$ (FC) without further requirements such as $\mathcal{P}$-sensitivity, see Example 3.4. Moreover, we illustrate that many conditions, in particular on $\mathcal{P}$, one would think of in the first place to ensure (WC) $\Leftrightarrow$ (FC), indeed imply that $\mathcal{P}$ is dominated, so we are back to Theroem 1.1. Hence, a further contribution of this paper is to provide a deeper insight into the fallacies one might encounter when attempting to extend Theorem 1.1 to a fully general robust case.

The paper is structured as follows: Section 2 provides a list of useful notations which will be used throughout the paper. Section 3 contains the main results of the paper, and in particular Theorem 3.9 is the robust version of Theorem 1.1. Finally, some applications of Theorem 3.9 in the field of Mathematical Finance are collected in Section 4. First, Theorem 4.3 provides a dual representation of convex and quasiconvex increasing functionals in this robust framework. Such representation results are key in the theory of convex risk measures. Secondly, Section 4.2 applies our results to reconcile, in this general robust setup, duality theory and the proof of the Fundamental Theorem of Asset Pricing. We do not assume that the reader is familiar with mathematical finance, and present the applications in a self contained way. But for the sake of brevity we do not explain the background of the applications and rather refer to the literature for this.

2 Notation

For the sake of clarity we propose here a list of the basic notations and definitions that we shall use throughout this paper.

Let $(\Omega, \mathcal{F})$ be any measurable space.

(i) $ba := \{\mu : \mathcal{F} \to \mathbb{R} \mid \mu \text{ is finitely additive}\}$ and $ca := \{\mu : \mathcal{F} \to \mathbb{R} \mid \mu \text{ is } \sigma\text{-additive}\}$. These are both Banach lattices once endowed with the total variation norm TV and $|\mu| = \mu^+ + \mu^-$ where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition (see [AB06] for further details).

(ii) ba_+ (resp. ca_+) is the set of all positive additive (resp. σ-additive) set functions on

$(\Omega, \mathcal{F})$.

(iii) In absence of any reference probability measure we have the following sets of random variables

$$\begin{aligned}
\mathcal{L} &:= \{f : \Omega \to \mathbb{R} \mid f \, \text{is} \, \mathcal{F}\text{-measurable}\}, \\
\mathcal{L}_+ &:= \{f \in \mathcal{L} \mid f(\omega) \geq 0 \ , \forall \, \omega \in \Omega\}, \\
\mathcal{L}^\infty &:= \{f \in \mathcal{L} \mid f \text{ is bounded}\}.
\end{aligned}$$

In particular $\mathcal{L}^\infty$ is a Banach space under the (pointwise) supremum norm $\|\cdot\|_\infty$ with dual space ba.

(iv) $\mathcal{M}_1 \subset ca_+$ is the set of all probability measures on $(\Omega, \mathcal{F})$.

(v) Throughout this paper we fix set of probability measures $\mathcal{P} \subset \mathcal{M}_1$.

(vi) We introduce the sublinear expectation

$$c(f) := \sup_{Q \in \mathcal{P}} E_Q[f], \quad f \in \mathcal{L}_+$$

and by some abuse of notation we set the capacity $c(A) := c(1_A)$ for $A \in \mathcal{F}$.

(vii) Let $\widehat{\mathcal{P}}, \widetilde{\mathcal{P}} \subseteq \mathcal{M}_1$. $\widehat{\mathcal{P}}$ dominates $\widetilde{\mathcal{P}}$, denoted by $\widetilde{\mathcal{P}} \ll \widehat{\mathcal{P}}$, if for all $A \in \mathcal{F}$:

$$\sup_{P \in \widehat{\mathcal{P}}} P(A) = 0 \quad \Rightarrow \quad \sup_{P \in \widetilde{\mathcal{P}}} P(A) = 0.$$

We say that two classes $\widehat{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ are equivalent, denoted by $\widehat{\mathcal{P}} \approx \widetilde{\mathcal{P}}$, if $\widetilde{\mathcal{P}} \ll \widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}} \ll \widetilde{\mathcal{P}}$.

(viii) A statement holds $\mathcal{P}$-quasi surely (q.s.) if the statement holds Q-almost surely (a.s.) for any $Q \in \mathcal{P}$.

(ix) The space of finitely additive (resp. countably additive) set functions dominated by c is given by $ba_c = \{\mu \in ba \mid \mu \ll c\}$ (resp. $ca_c = \{\mu \in ca \mid \mu \ll c\}$). Here $\mu \ll c$ means: $c(A) = 0$ for some $A \in \mathcal{F}$ implies $\mu(A) = 0$.
When $\mathcal{P} = \{Q\}$ we shall write ba_Q or ca_Q for sake of simplicity.

(x) We consider the quotient space $L_c := \mathcal{L}_{/\sim}$ where the equivalence is given by

$$f \sim g \quad \Leftrightarrow \quad \forall P \in \mathcal{P} : P(f = g) = 1.$$

We shall use capital letters to distinguish equivalence classes of random variables $X \in L_c$ from a representative $f \in X$, with $f \in \mathcal{L}$.

(xi) For any $f, g \in \mathcal{L}$ and $P \in \mathcal{M}_1$, we write $f \leq g$ P-a.s. if and only if $P(f \leq g) = 1$. Similarly $f \leq g$ $\mathcal{P}$-q.s. if and only if $f \leq g$ P-a.s. for all $P \in \mathcal{P}$. This relation is a partial order on $\mathcal{L}$ and thus also on L where $X \leq Y$ $\mathcal{P}$-q.s. if and only if $f \leq g$ $\mathcal{P}$-q.s. for any $f \in X$ and $g \in Y$.

(xii) We set $L_c^\infty := \mathcal{L}^\infty_{/\sim}$ and endow it with the norm

$$\|X\|_{c,\infty} := \inf\{m \mid \forall P \in \mathcal{P} : P(|X| \leq m) = 1\}.$$

$(L_c^\infty, \|\cdot\|_{c,\infty})$ is a Banach lattice with partial order $\leq$ $\mathcal{P}$-q.s. Its norm dual is ba_c. In case $\mathcal{P} = \{Q\}$ we shall write L_Q^∞ and $\|\cdot\|_{Q,\infty}$ for the sake of simplicity .

3 Towards a robust version of Theorem 1.1

We start by recalling the proof of Theorem 1.1: the idea is to apply the Krein-Smulian theorem (see [FS04, Theorem A.64]) which implies that we only need to show that the sets

$$C_K := \mathcal{A} \cap \{X \in L_P^\infty \mid \|X\|_{P,\infty} \leq K\}$$

are weak*-closed for any constant $K > 0$. Note that the inclusion

$$i : (L_P^\infty, \sigma(L_P^\infty, L_P^1)) \to (L_P^1, \sigma(L_P^1, L_P^\infty)) \tag{3.1}$$

is continuous. Now, as $\mathcal{A}$ is Fatou closed, i.e. closed under bounded P-a.s. convergence, it follows that $i(C_K)$ is a closed subset of the Banach space $(L_P^1, E_P[|\cdot|])$, and thus $i(C_K)$ is also weakly (i.e. $\sigma(L_P^1, L_P^\infty)$) closed by convexity, so eventually C_K must be weak*-closed by continuity of i.

Therefore a natural approach to prove a robust version of Theorem 1.1 is to 'robustify' the spaces L_P^1 and try to repeat the argument above. There are two natural candidates for this: Let $H_c := \{X \in L \mid c(|X|) < \infty\}$, with norm $\|X\|_c := c(|X|)$. Then it is readily verified that $(H_c, \|\cdot\|_c)$ is a Banach lattice. But in the robust case there is also another candidate, namely $M_c := \overline{L_c^\infty}^{\|\cdot\|_c}$ which is also a Banach lattice with the norm $\|\cdot\|_c$. These spaces have recently been studied in the literature, see e.g. [DHP11] and [Nu14], since they appear as natural environments to embed financial modelling under uncertainty. Clearly, $L_c^\infty \subset M_c \subset H_c \subset L$. Note that the trick with the inclusion (3.1) requires that the norm dual of L_P^1 can be identified with L_P^∞, so in particular with a subset of L_P^1 where in this latter case L_P^1 is viewed as a representation of ca_P. Thus the reader may readily check that we could save the above argument if the norm duals M_c^* and H_c^* of M_c and H_c, respectively, would satisfy $M_c^* \subset ca$ or $H_c^* \subset ca$. The following Theorem 3.1 shows that this is the case only if $\mathcal{P}$ is dominated. To this end, denote by

$$\mathcal{Z} := \{(A_n)_{n\in\mathbb{N}} \subset \mathcal{F} \mid A_n \downarrow \emptyset \text{ and } c(A_n) \not\to 0\}, \tag{3.2}$$

where $A_n \downarrow \emptyset$ means that $A_n \supset A_{n+1}$, $A_n \neq \emptyset$, $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, the decreasing sequences of sets on which c is not continuous.

Theorem 3.1. *Consider the following conditions:*

(i) $\mathcal{Z} = \emptyset$.

(ii) $M_c^* \subset ca$.

(iii) $H_c^* \subset ca$.

Then (i) $\iff$ *(ii)* $\Longleftarrow$ *(iii).*

In particular, if $\mathcal{Z} = \emptyset$*, then there exists a countable subset* $\widetilde{\mathcal{P}} \subset \mathcal{P}$ *such that* $\widetilde{\mathcal{P}} \approx \mathcal{P}$*, and thus there is a probability measure* $Q \in \mathcal{M}_1$ *such that* $\{Q\} \approx \mathcal{P}$*.*

Proof. (i) $\Rightarrow$ (ii): By Proposition A.2 for any $l \in M_c^*$ there is $\mu \in ca$ such that $l(X) = \int X\, d\mu$ for all simple random variables X. Moreover, $\mu \in ca_c$, because $c(A) = 0$ implies $l(1_A) = 0$, $A \in \mathcal{F}$. Since for any $X \in L_c^\infty$ and any $n \in \mathbb{N}$ by the usual approximation method from integration theory there is a simple random variable X_n such that $|X - X_n| < 1/n$ $\mathcal{P}$-q.s., so $\|X - X_n\|_c < 1/n$, continuity of l and the dominated convergence theorem yield

$$l(X) = \lim_{n\to\infty} l(X_n) = \lim_{n\to\infty} \int X_n\, d\mu = \int X\, d\mu$$

for all $X \in L_c^\infty$. We recall that in [DHP11] Proposition 18 the following relation was shown

$$M_c = \{X \in H_c \mid \lim_{n\to\infty} \|X 1_{\{|X|\geq n\}}\|_c = 0\}.$$

Hence, for $X \in (M_c)_+$ we have by monotone convergence that

$$l(X) = \lim_{n\to\infty} l(X 1_{\{|X|\leq n\}}) = \lim_{n\to\infty} \int X 1_{\{|X|\leq n\}}\, d\mu = \int X\, d\mu.$$

Finally, decomposing $X \in M_c$ into $X^+ - X^-$ with $X^+, X^- \in (M_c)_+$ and linearity of l and the integral shows (ii).

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) follow directly from Proposition A.2

The last statement of this theorem is Proposition A.1. □

Remark 3.2. *Note that the converse of the last statement of Theorem 3.1 is not true, i.e.* $\mathcal{Z} \neq \emptyset$ *does not imply that* $\mathcal{P}$ *is not dominated. To see this, let* $A_n \downarrow \emptyset$ *and pick a sequence of probability measures* P_n *such that* $P_n(A_n) = 1$ *for all* $n \in \mathbb{N}$*, and let* $\mathcal{P} = \{P_n \mid n \in \mathbb{N}\}$*. Then, clearly* $\|1_{A_n}\|_c = 1$ *for each* n*. Hence,* $\mathcal{Z} \neq \emptyset$ *and thus* $M_c^* \not\subset ca$*. However, we have that* $\{Q\} \approx \mathcal{P}$ *for* $Q = \sum_{n=1}^\infty \frac{1}{2^n} P_n$*.*

Recall the conditions

(WC) $\mathcal{A} \subset L_c^\infty$ is $\sigma(L_c^\infty, ca_c)$-closed.

(FC) $\mathcal{A} \subset L_c^\infty$ is Fatou closed: for any bounded sequence $X_n \subset \mathcal{A}$ and $X \in L_c^\infty$ such that $X_n \to X$ $\mathcal{P}$-q.s. we have that $X \in \mathcal{A}$.

It is easily verified that always (WC) $\Longrightarrow$ (FC) since any bounded $\mathcal{P}$-q.s. converging sequence also converges in $\sigma(L_c^\infty, ca_c)$ to the same limit. However, there is in general no proof of (FC) $\Longrightarrow$ (WC) even if $\mathcal{A}$ is convex, and also requiring monotonicity of $\mathcal{A}$, i.e. $\mathcal{A} + (L_c^\infty)_+ = \mathcal{A}$, in addition is not sufficient:

Theorem 3.3. *Let $\mathcal{A} \subset L_c^\infty$ be convex and monotone. Without further assumptions on $\mathcal{P}$ or $\mathcal{A}$, there exists no proof of (FC) $\Rightarrow$ (WC).*

The proof of Theorem 3.3 is given by the following Example 3.4 where we give a counter-example of (FC) $\Longrightarrow$ (WC) assuming the continuum hypothesis. So under the continuum hypothesis (FC) $\Longrightarrow$ (WC) is indeed wrong. Note that as the continuum hypothesis does not conflict with what one perceives as standard mathematical axioms, there is of course no way to prove (FC) $\Longrightarrow$ (WC) even if we do not believe in the continuum hypothesis.

Example 3.4. *Consider the measure space $(\Omega, \mathcal{F}) = ([0,1], \mathfrak{P}([0,1]))$, where $\mathfrak{P}([0,1])$ denotes the power set of $[0,1]$. Assume the continuum hypothesis. Banach and Kuratowski have shown that for any set I with the same cardinality as $\mathbb{R}$ there is no measure μ on $(I, \mathfrak{P}(I))$ such that $\mu(I) = 1$ and $\mu(\{\omega\}) = 0$ for all $\omega \in I$; see for instance [Du02, Theorem C.1]. It follows that any probability measure μ over $(\Omega, \mathcal{F})$ must be a countable sum of weighted Dirac-measures, i.e. $\mu = \sum_{i=1}^\infty a_i \delta_{\omega_i}$ where $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, $\omega_i \in \Omega$, $i \in \mathbb{N}$. (Recall that for $\omega \in \Omega$ and $A \in \mathcal{F}$: $\delta_\omega(A) = 1$ if and only if $\omega \in A$ and $\delta_\omega(A) = 0$ otherwise.) Indeed, let $\mu \in \mathcal{M}_1$, and let*

$$S := \{\omega \in \Omega \mid \mu(\{\omega\}) > 0\}.$$

Then S can at most be countable (consider the sets $S_n := \{\omega \in \Omega \mid \mu(\{\omega\}) > 1/n\}$, $n \in \mathbb{N}$, and note that $S = \bigcup_{n \in \mathbb{N}} S_n$). Now suppose that $\mu([0,1] \setminus S) > 0$, then as $[0,1] \setminus S$ has the same cardinality as $[0,1]$, this implies the existence of an atom for the measure μ restricted to $[0,1] \setminus S$, i.e. there exists $\hat{\omega} \in [0,1] \setminus S$ such that

$$\frac{1}{\mu([0,1] \setminus S)} \mu(\{\hat{\omega}\}) > 0.$$

This clearly contradicts the definition of S.

Let $\mathcal{P} := \{\delta_\omega \mid \omega \in [0,1]\}$ be the set of all Dirac measures. Then

$$c(|X|) = \sup_{\omega \in [0,1]} |X(\omega)|,$$

so it turns out that $L_c^\infty = M_c = H_c = \mathcal{L}^\infty$. *Hence,* $(L_c^\infty)^* = M_c^* = H_c^* = ba$, *and, as* $c(A) = 0$ *is equivalent to* $A = \emptyset$, *we also have that* $ca_c = ca$. *Consider the set*

$$C := \{1_A \mid \emptyset \neq A \subset [0,1] \text{ is countable}\},$$

and let $\mathcal{A}$ *be the convex closure of* C *under bounded* $\mathcal{P}$*-q.s. convergence. Then* $1 \notin \mathcal{A}$*: Indeed, any* $X = \sum_{i=1}^n a_i 1_{A_i}$, $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, $1_{A_i} \in C$, *in the convex hull of* C *satisfies* $0 \leq X \leq 1_{A_X}$ *where* $A_X := \bigcup_{i=1}^n A_i$ *is countable. Let* X_k *be any sequence in the convex hull of* C, *then* $0 \leq X_k \leq 1_B$, $k \in \mathbb{N}$, *where* $B := \bigcup_{k\in\mathbb{N}} A_{X_k}$ *is countable. Hence,* $X_k(\omega) = 0$ *for all* $\omega \in [0,1] \setminus B$, *so* $1 \notin \mathcal{A}$. *Now consider the family* $\mathcal{G}$ *of all countable subsets of* $[0,1]$ *directed by* $A \leq B$ *if and only if* $A \subset B$. *Consider the net* $\{1_A \mid A \in \mathcal{G}\} \subset C$. *Then for any probability measure* μ *there is* $A \in \mathcal{G}$ *(namely* $A = S$*) such that for all* $B \in \mathcal{G}$ *with* $B \geq A$ *we have* $\int 1_B \, d\mu = 1 = \int 1 \, d\mu$. *Thus* 1 *lies in the* $\sigma(L_c^\infty, ca_c)$*-closure of* $\mathcal{A}$.

In order to make the presentation simpler, we did not require monotonicity of $\mathcal{A}$ *so far, but the same arguments as above show that if* $\mathcal{A}$ *is the convex closure of* $-C + (L_c^\infty)_+$ *under bounded* $\mathcal{P}$*-q.s. convergence, which is convex and monotone, then* $-1 \notin \mathcal{A}$ *but* -1 *is an element of the* $\sigma(L_c^\infty, ca_c)$*-closure of* $\mathcal{A}$.

A consequence of Theorem 3.3 is that we need to ask for additional properties on $\mathcal{A}$ in order to have (FC) $\iff$ (WC). A property which solves the problem is the so-called $\mathcal{P}$-sensitivity discussed in the following section.

3.1 $\mathcal{P}$-sensitivity, $ca_c^* = L_c^\infty$, and (FC) $\iff$ (WC)

A simple property on $\mathcal{A}$ which allows to prove (FC) $\iff$ (WC) is to require that the convex set $\mathcal{A} \subset L_c^\infty$ behaves as in the dominated case, i.e. there is a reference probability $P \in \mathcal{P}$ such that $\mathcal{A}$ is closed under bounded P-a.s. convergence. Under this assumption the whole issue can be reduced to Theorem 1.1. Clearly, this assumption is too strong. However, it gives the idea of the $\mathcal{P}$-sensitivity property we will introduce in the following.

Given a probability $Q \in \mathcal{M}_1$ such that $\{Q\} \ll \mathcal{P}$ we define the linear map $j_Q : L_c^\infty \to L_Q^\infty$ by $Q(j_Q(X) = X) = 1$, i.e. $j_Q(X)$ is the equivalence class in L_Q^∞ such that any representative of $j_Q(X)$ and any representative of X are Q-a.s. identical. As ca_Q (which can be identified with L_Q^1) is a subset of ca_c, we deduce that $j_Q : (L_c^\infty, \sigma(L_c^\infty, ca_c)) \to (L_Q^\infty, \sigma(L_Q^\infty, L_Q^1))$ is continuous.

Definition 3.5. *A set* $\mathcal{A} \subseteq L_c^\infty$ *is called* $\mathcal{P}$-sensitive *if there exists a set* $\mathcal{Q} \subset \mathcal{M}_1$ *with* $\mathcal{Q} \ll \mathcal{P}$ *such that*

$$j_Q(X) \in j_Q(\mathcal{A}) \text{ for all } Q \in \mathcal{Q} \text{ implies } X \in \mathcal{A}$$

or equivalently

$$\mathcal{A} = \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}).$$

The set $\mathcal{Q}$ *will be called* reduction set *for* $(\mathcal{A}, \mathcal{P})$.

Remark 3.6. *Suppose that* $\mathcal{P}$ *is dominated. Then the Halmos Savage lemma (see [HS49], Lemma 7) guarantees the existence of a countable subclass* $\{P_i\}_{i=1}^{\infty}$ *such that* $\{P_i\}_{i=1}^{\infty} \approx \mathcal{P}$. *Let* $P = \sum \frac{1}{2^i} P_i$. *Then* $\mathcal{P} \approx \{P\}$, *so the space* L_c^∞ *can be identified with* L_P^∞. *Hence, in that case any set* $\mathcal{A} \subseteq L_c^\infty$ *is automatically* $\mathcal{P}$*-sensitive with reduction set* $\mathcal{Q} = \{P\}$.

Example 3.7. *The set* $\mathcal{A}$ *of Example 3.4 is not* $\mathcal{P}$*-sensitive. Since* $c(A) = 0$ *implies that* $A = \emptyset$, *any set of probabilities* $\mathcal{Q} \subset \mathcal{P}$ *satisfies* $\mathcal{Q} \ll \mathcal{P}$. *Let* $Q \in \mathcal{M}_1$ *be arbitrary and* $S := \{\omega \in [0,1] \mid Q(\{\omega\}) > 0\}$ *such that* $Q = \sum_{\omega \in S} a_\omega \delta_\omega$ *with* $a_\omega > 0$ *and* $\sum_{\omega \in S} a_\omega = 1$. *Then* $1_S \in \mathcal{A}$ *by definition of* $\mathcal{A}$ *and thus* $1 \in j_Q(\mathcal{A})$, *or to be more precise,* 1 *and* 1_S *form the same equivalence class in* L_Q^∞. *Since* $Q \in \mathcal{M}_1$ *was arbitrary, we have* $1 \in \bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A})$. *As we know that* $1 \notin \mathcal{A}$, *the set* $\mathcal{A}$ *is not* $\mathcal{P}$*-sensitive.*

Indeed $\mathcal{P}$-sensitivity is a necessary condition for (FC) $\iff$ (WC).

Proposition 3.8. *Any convex set* $\mathcal{A} \subset L_c^\infty$ *which is* $\sigma(L_c^\infty, ca_c)$*-closed (i.e. satisfies (WC)) is* $\mathcal{P}$*-sensitive.*

Proof. If $\mathcal{A} = \emptyset$ or $\mathcal{A} = L_c^\infty$, the assertion is trivial. Now assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq L_c^\infty$. As $\mathcal{A}$ is $\sigma(L_c^\infty, ca_c)$-closed and convex, the function

$$\rho(X) := \delta(X \mid \mathcal{A}) := \begin{cases} 0 & \text{if } X \in \mathcal{A} \\ \infty & \text{else} \end{cases}, \quad X \in L_c^\infty,$$

is convex and $\sigma(L_c^\infty, ca_c)$ lower-semicontinuous. Hence, by the Fenchel-Moreau theorem (see [ET99, Proposition 4.1]) there exists a dual representation of ρ, i.e.

$$\rho(X) = \sup_{\mu \in \mathcal{Q}} \left\{ \int X \, d\mu - \rho^*(\mu) \right\}$$

where $\mathcal{Q} := \{\mu \in ca_c \mid \rho^*(\mu) < \infty\}$ is a convex set and

$$\rho^*(\mu) := \sup_{X \in \mathcal{A}} \int X \, d\mu, \quad \mu \in ca_c.$$

$\mathcal{A} \neq L_c^\infty$ implies $\mathcal{Q} \supsetneqq \{0\}$ and therefore,

$$\mathcal{A} = \bigcap_{\mu \in \mathcal{Q}} \left\{ X \in L_c^\infty \mid \int X \, d\mu \leq \rho^*(\mu) \right\} = \bigcap_{\mu \in \mathcal{Q} \setminus \{0\}} \left\{ X \in L_c^\infty \mid \int X \, d\mu \leq \rho^*(\mu) \right\}.$$

Let $\tilde{\mathcal{Q}} := \{\frac{|\mu|}{|\mu|(\Omega)} \mid \mu \in \mathcal{Q} \setminus \{0\}\} \subset \mathcal{M}_1$ and note that $\tilde{\mathcal{Q}} \ll \mathcal{P}$ since $\mathcal{Q} \subset ca_c$. Consider

$$X \in \bigcap_{Q \in \tilde{\mathcal{Q}}} j_Q^{-1} \circ j_Q(\mathcal{A}).$$

Fix $Q \in \tilde{\mathcal{Q}}$ and $\nu \in \mathcal{Q}$ such that $Q = \frac{|\nu|}{|\nu|(\Omega)}$. Then, $j_Q(X) \in j_Q(\mathcal{A})$, i.e. there is $Y \in \mathcal{A}$ such that $j_Q(X) = j_Q(Y)$. Noting that $X = j_Q(X)$ and $Y = j_Q(Y)$ under ν, it follows that

$$\int X\, d\nu = \int j_Q(X)\, d\nu = \int j_Q(Y)\, d\nu = \int Y\, d\nu \leq \rho^*(\nu),$$

where the inequality follows from $Y \in \mathcal{A}$. Since $Q \in \tilde{\mathcal{Q}}$ was arbitrary, we conclude that indeed $\int X\, d\mu \leq \rho^*(\mu)$ for all $\mu \in \mathcal{Q}$, and hence that $X \in \mathcal{A}$. This shows that $\bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}) \subset \mathcal{A}$. The other inclusion $\bigcap_{Q \in \mathcal{Q}} j_Q^{-1} \circ j_Q(\mathcal{A}) \supset \mathcal{A}$ is trivially satisfied, so we have that $\mathcal{A}$ is $\mathcal{P}$-sensitive with reduction set $\tilde{\mathcal{Q}}$. □

The following Theorem 3.9 gives conditions under which (FC) $\iff$ (WC) for a convex set $\mathcal{A} \subset L_c^\infty$. Besides $\mathcal{P}$-sensitivity we have to require that the norm dual ca_c^* of (ca_c, TV), where TV denotes the total variation norm on ca_c, may be identified with L_c^∞. Clearly any $X \in L_c^\infty$ may be identified with a continuous linear functional on ca_c by

$$ca_c \ni \mu \mapsto \int X\, d\mu, \tag{3.3}$$

so we always have $L_c^\infty \subset ca_c^*$. However, $ca_c^* = L_c^\infty$ is obviously a very strong condition which we will characterize in Proposition 3.10 in terms of the existence of the essential supremum in the $\mathcal{P}$-quasi sure sense.

Theorem 3.9. *Suppose that $ca_c^* = L_c^\infty$ and let $\mathcal{A} \subset L_c^\infty$ be convex and monotone ($\mathcal{A} + (L_c^\infty)_+ = \mathcal{A}$). Equivalent are*

(i) *$\mathcal{A}$ satisfies (WC).*

(ii) *$\mathcal{A}$ is $\mathcal{P}$-sensitive and satisfies (FC).*

Proof. We already know that (WC) implies (FC) and $\mathcal{P}$-sensitivity. Now assume that $\mathcal{A}$ is $\mathcal{P}$ sensitive and satisfies (FC). Since $ca_c^* = L_c^\infty$, by the Krein-Smulian theorem (see [FS04, Theorem A.64]) it is sufficient to show that $C_K := \mathcal{A} \cap \{Z \in L^\infty \mid \|Z\|_{c,\infty} \leq K\}$ is $\sigma(L_c^\infty, ca_c)$-closed for every $K > 0$. Let $\mathcal{Q}$ be a reduction set for $(\mathcal{A}, \mathcal{P})$ and fix any $K > 0$ and $Q \in \mathcal{Q}$.

Consider the continuous inclusion

$$i : (L_Q^\infty, \sigma(L_Q^\infty, L_Q^1)) \to (L_Q^1, \sigma(L_Q^1, L_Q^\infty)).$$

In a first step we show that $C_{Q,K} := i \circ j_Q(C_K)$ is $\|\cdot\|_Q := E_Q[|\cdot|]$-closed in L_Q^1, because being convex it then follows that $C_{Q,K}$ is $\sigma(L_Q^1, L_Q^\infty)$-closed and therefore $j_Q(C_K)$

is $\sigma(L^\infty_Q, L^1_Q)$-closed by continuity of i. To this end let $(Y_n)_{n\in\mathbb{N}} \subset C_{Q,K}$ and $Y \in L^1_Q$ such that $\|Y_n - Y\|_Q \to 0$, and without loss of generality we may also assume that $Y_n \to Y$ Q-a.s. Choose $X_n \in C_K$ such that $Y_n = j_Q(X_n)$ for all $n \in \mathbb{N}$ and $X \in L^\infty_c$ such that $Y = j_Q(X)$. Consider now the set

$$F := \{\omega \in \Omega \mid X_n(\omega) \to X(\omega)\}$$

(by the usual abuse of notation, in the definition of F we still write X_n and X for arbitrary representatives of the equivalence classes X_n and X). By monotonicity of $\mathcal{A}$ we have that $\widetilde{X}_n := X_n 1_F + K 1_{F^c} \in C_K$ for all $n \in \mathbb{N}$, and $\widetilde{X}_n \to X 1_F + K 1_{F^c} =: \widetilde{X}$ $\mathcal{P}$-q.s. Consequently $\widetilde{X} \in C_K$ and since $Q(F) = 1$ we have $Y = j_Q(X) = j_Q(\widetilde{X}) \in C_{Q,K}$. Hence, $j_Q(C_K)$ is $\sigma(L^\infty_Q, L^1_Q)$ closed.

By continuity of j_Q, the preimage $j_Q^{-1} \circ j_Q(C_K)$ is $\sigma(L^\infty_c, ca_c)$-closed, and as also $\{X \mid \|X\|_{c,\infty} \le K\}$ is $\sigma(L^\infty_c, ca_c)$-closed, we conclude that

$$A_{Q,K} := j_Q^{-1} \circ j_Q(C_K) \cap \{X \mid \|X\|_{c,\infty} \le K\} \supseteq C_K$$

and finally also $\bigcap_{Q\in\mathcal{Q}} A_{Q,K}$ are $\sigma(L^\infty_c, ca_c)$-closed. Clearly, $\bigcap_{Q\in\mathcal{Q}} A_{K,Q} \supseteq C_K$. If we can show $\bigcap_{Q\in\mathcal{Q}} A_{Q,K} \subseteq C_K$, then we are done, because then $\bigcap_{Q\in\mathcal{Q}} A_{Q,K} = C_K$, and thus C_K is $\sigma(L^\infty_c, ca_c)$-closed. To this end, let $X \in \bigcap_{Q\in\mathcal{Q}} A_{Q,K}$. Then $j_Q(X) \in j_Q(\mathcal{A})$ for any $Q \in \mathcal{Q}$ and therefore $X \in \mathcal{A}$ by $\mathcal{P}$-sensitivity. Moreover by definition of $A_{K,Q}$ we also have $\|X\|_{c,\infty} \le K$. □

Let $\mathcal{D} \subset L^\infty_c$. An essential supremum of $\mathcal{D}$ is a least upper bound of $\mathcal{D}$, that is an $X \in L^\infty_c$ such that $Y \le X$ $\mathcal{P}$-q.s. for all $Y \in \mathcal{D}$, and any $Z \in L^\infty_c$ such $Y \le Z$ $\mathcal{P}$-q.s. for all $Y \in \mathcal{D}$ satisfies $X \le Z$ $\mathcal{P}$-q.s. The essential supremum of $\mathcal{D}$ is denoted by $\operatorname{ess\,sup}_{Y\in\mathcal{D}} Y$. Similarly an essential infimum denoted by $\operatorname{ess\,inf}_{Y\in\mathcal{D}} Y$ of $\mathcal{D}$ is a greatest lower bound of $\mathcal{D}$, i.e. an essential supremum of $-\mathcal{D}$.

Proposition 3.10. *$ca^*_c = L^\infty_c$ if and only if there exists an essential supremum for any norm bounded set $\mathcal{D} \subset L^\infty_c$.*

Proof. We first prove that the existence of an essential supremum implies $ca^*_c = L^\infty_c$ in four steps. To this end fix $l \in ca^*_c$, and without loss of generality we may assume that the operator norm $\|l\|^*$ of l satisfies $\|l\|^* \le 1$.

Step 1: Let $\mu \in (ca_c)_+$. Recall that ca_μ, i.e. the space of measures ν on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$, is a subset of ca_c which may be identified with L^1_μ. Hence, l restricted to ca_μ may be seen as a continuous linear functional on L^1_μ, the dual of which may be identified with L^∞_μ. Thus there exists an element in L^∞_μ and therefore also some $X_\mu \in L^\infty_c$ such that $l(\nu) = \int X_\mu \, d\nu$ for all $\nu \in ca_\mu$. Notice that for any $A \in \mathcal{F}$, $\int_A X_\mu \, d\nu = l(1_A \nu)$ where

$1_A\,\nu(\cdot) = \nu(\cdot \cap A) \in ca_c$. In particular, we can assume that $\|X_\mu\|_{c,\infty} \le 1$ for all $\mu \in (ca_c)_+$, because for any $A \in \mathcal{F}$ with $\mu(A) > 0$ we have that

$$\int_A X_\mu \, d\mu = l(1_A\,\mu) \le \|l\|^* TV(1_A\,\mu) \le \mu(A),$$

so $X_\mu \le 1$ μ-a.s., and thus we can exchange X_μ with $X_\mu 1_{\{X_\mu \le 1\}}$.

Step 2: Let $\mu, \nu \in (ca_c)_+$ such that $\nu \ll \mu$. Since $ca_\nu \subset ca_\mu$ it follows that $X_\nu = X_\mu$ ν-a.s. (more precisely: $f = g$ ν-a.s. for all $f \in X_\nu$ and $g \in X_\mu$)

Step 3: Consider the family $\mathcal{D} := \{X_\mu \mid \mu \in (ca_c)_+\}$. As $\mathcal{D}$ and any subfamilies are norm bounded by step 1, the essential suprema and essential infima of these families exist. This implies the existence of

$$\liminf_\mu X_\mu := \operatorname*{ess\,sup}_{\nu \in (ca_c)_+} \operatorname*{ess\,inf}_{\nu \ll \mu} X_\mu$$

as an essential supremum of essential infima, and also of

$$\limsup_\mu X_\mu := \operatorname*{ess\,inf}_{\nu \in (ca_c)_+} \operatorname*{ess\,sup}_{\nu \ll \mu} X_\mu.$$

As for any $\mu, \nu \in (ca_c)_+$ there is $\zeta \in (ca_c)_+$ such that $\mu \ll \zeta$ and $\nu \ll \zeta$ (e.g. $\zeta = \frac{1}{2}(\mu + \nu)$) we conclude that indeed $\liminf_\mu X_\mu \le \limsup_\mu X_\mu$.

Step 4: For any $\mu \in (ca_c)_+$ we compute

$$l(\mu) = \int X_\mu \, d\mu = \int \inf_{\mu \ll \nu} X_\nu \, d\mu \le \int \liminf_\nu X_\nu \, d\mu,$$

where we used $\mu \in ca_\mu$ in the first equality and step 2 in the second. Similarly

$$l(\mu) = \int X_\mu \, d\mu = \int \sup_{\mu \ll \nu} X_\nu \, d\mu \ge \int \limsup_\nu X_\nu \, d\mu.$$

As $\liminf_\mu X_\mu \le \limsup_\mu X_\mu$, we must have

$$l(\mu) = \int \liminf_\nu X_\nu \, d\mu = \int \limsup_\nu X_\nu \, d\mu, \quad \text{for all } \mu \in (ca_c)_+.$$

By linearity of l we conclude that indeed

$$l(\mu) = \int \liminf_\nu X_\nu \, d\mu = \int \limsup_\nu X_\nu \, d\mu, \quad \text{for all } \mu \in ca_c,$$

so eventually l may be identified with $Y := \liminf_\nu X_\nu = \limsup_\nu X_\nu \in L_c^\infty$. This proves $ca_c^* = L_c^\infty$.

In order to prove that $ca_c^* = L_c^\infty$ implies the existence of an essential supremum for any norm bounded set $\mathcal{D} \subset L_c^\infty$, we recall that ca and thus also ca_c is an so–called AL-space ([AB06, Theorem 10.56]), so ca_c^* is an AM-space ([AB06, Theorem 9.27]). In particular ca_c^* is order complete, that means that for any subset of ca_c^* which is order bounded from above there exists a least upper bound. Here, the order $\geq_*$ on ca_c^* is given by $l \geq_* 0$ if and only if $l(\mu) \geq 0$ for all $\mu \in (ca_c)_+$, and a set $\mathcal{S} \subset ca_c^*$ is order bounded from above if there is $h \in ca_c^*$ such that $h - l \geq_* 0$ for all $l \in \mathcal{S}$; for a survey of ordered spaces we refer to [AB06]. Any norm bounded $\mathcal{D} \subset L_c^\infty$ is order bounded from above in ca_c^*, because $K\mu(\Omega) - \int X \, d\mu \geq 0$, $\mu \in (ca_c)_+$, for a constant $K > 0$ which is an upper bound of the norm on $\mathcal{D}$, so $(\mu \mapsto K\mu(\Omega)) \in ca_c^*$ is an upper bound with respect to $\geq_*$. Thus there is a least upper bound of $\mathcal{D}$ viewed as a subset of ca_c^*. Now suppose that ca_c^* can be identified with L_c^∞. Then this least upper bound of $\mathcal{D}$ may be identified with an element in $X \in L_c^\infty$, that is

$$\int X \, d\mu \geq \int Y \, d\mu \quad \text{for all } \mu \in (ca_c)_+ \text{ and all } Y \in \mathcal{D}.$$

Considering measures μ of type $1_A dP$ for $P \in \mathcal{P}$ and $A \in \mathcal{F}$ shows that $X \geq Y$ $\mathcal{P}$-q.s. for all $Y \in \mathcal{D}$, and $\mu \mapsto \int X \, d\mu$ being the least amongst the upper bounds of $\mathcal{D}$ in the $\geq_*$-order implies that X is an essential supremum of $\mathcal{D}$. □

Example 3.11. *In this example we fix a measure space $(\Omega, \mathcal{F})$ and an uncountable family $\mathcal{P} = \{P_\sigma\}_{\sigma \in \Sigma}$ of probability measures. Consider the enlarged sigma algebra $\mathcal{F}^\Sigma = \bigcap_{\sigma \in \Sigma} \mathcal{F}^\sigma$ where $\mathcal{F}^\sigma$ is the P^σ completion of $\mathcal{F}$, and notice that any P^σ uniquely extends to $\mathcal{F}^\Sigma$. Assume that there exists a family of sets $\{\Omega^\sigma\}_{\sigma \in \Sigma} \subseteq \mathcal{F}^\Sigma$ such that for any $\sigma \in \Sigma$, $P^\sigma(\Omega^\sigma) = 1$ and $P^{\tilde{\sigma}}(\Omega^\sigma) = 0$ for $\tilde{\sigma} \neq \sigma$. In this case it is easily seen that any norm bounded set $\mathcal{D} \subset L_c^\infty(\Omega, \mathcal{F}^\Sigma)$ admits an essential supremum given by*

$$\operatorname{ess\,sup}_{Y \in \mathcal{D}} Y = \sum_{\sigma \in \Sigma} j_{P^\sigma}^{-1}(\operatorname{ess\,sup}_{Y \in \mathcal{D}} j_{P^\sigma}(Y)) 1_{\Omega^\sigma}.$$

Note that $\operatorname{ess\,sup}_{Y \in \mathcal{D}} j_{P^\sigma}(Y)$ in $L_{P^\sigma}^\infty$ is well-defined for every $\sigma \in \Sigma$. Also notice that $\operatorname{ess\,sup}_{Y \in \mathcal{D}} Y$ is $\mathcal{F}^\sigma$-measurable for any $\sigma \in \Sigma$ and therefore is also $\mathcal{F}^\Sigma$-measurable. Therefore $L_c^\infty(\Omega, \mathcal{F}^\Sigma) = ca_c^*(\Omega, \mathcal{F}^\Sigma)$. We refer to [Co12] for a deeper study of this example and applications to mathematical finance.

Example 3.12. *Recall Example 3.4. Clearly any norm bounded set $\mathcal{D} \subset L_c^\infty = \mathcal{L}^\infty$ admits an essential supremum which is simply given by $\omega \mapsto \sup_{Y \in \mathcal{D}} Y(\omega)$. Hence $ca^* = ca_c^* = \mathcal{L}^\infty$ by Proposition 3.10. This holds without the continuum hypothesis, but is also easily directly verified now using the continuum hypothesis: Let $l \in ca_c^*$ and define $X(\omega) = l(\delta_\omega)$, $\omega \in [0,1]$. Then by linearity, for all $\mu \in ca$ it follows that $l(\mu) = \sum_{\omega \in S} a_\omega l(\delta_\omega) = \int X \, d\mu$ where $S := \{\omega \in [0,1] \mid \mu(\{\omega\}) > 0\}$ and $a_\omega = \mu(\{\omega\})$, $\omega \in S$.*

4 Applications of Theorem 3.9

4.1 Dual representation of (quasi)convex increasing functionals

Definition 4.1. *A function $f : L_c^\infty \to (-\infty, \infty]$ is*

- *quasiconvex (resp. convex) if for every* $\lambda \in [0,1]$ *and* $X, Y \in L^\infty$ *we have* $f(\lambda X + (1-\lambda)Y) \leq \max\{X, Y\}$ *(resp.* $f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda)f(Y)$*).*

- τ*-lower semicontinuous (l.s.c.) for some topology* τ *on* L_c^∞ *if for every* $a \in \mathbb{R}$ *the lower level set* $\{X \in L_c^\infty \mid f(X) \leq a\}$ *is* τ*-closed.*

- $\mathcal{P}$*-sensitive if the lower level sets* $\{X \in L_c^\infty \mid f(X) \leq a\}$ *are* $\mathcal{P}$*-sensitive for every* $a \in \mathbb{R}$*.*

The following Lemma provides a huge class of $\mathcal{P}$-sensitive functions.

Lemma 4.2. *Consider a function* $f : L_c^\infty \to [-\infty, \infty]$ *such that*

$$f(X) = \sup_{P \in \mathcal{Q}} f_P(j_P(X)), \tag{4.4}$$

for some $\mathcal{Q} \subset \mathcal{M}_1$ *and* $f_P : L_P^\infty \to [-\infty, \infty]$*. If* $\mathcal{Q} \ll \mathcal{P}$ *then* f *is* $\mathcal{P}$*-sensitive with reduction set* $\mathcal{Q}$*.*

Proof. From representation (4.4) we automatically have

$$\{X \in L_c^\infty \mid f(X) \leq a\} = \bigcap_{P \in \mathcal{Q}} \{X \in L_c^\infty \mid f_P(j_P(X)) \leq a\}.$$

As $\{X \in L_c^\infty \mid f_P(j_P(X)) \leq a\} = j_P^{-1} \circ j_P\{X \in L_c^\infty \mid f_P(j_P(X)) \leq a\}$, we conclude that f is $\mathcal{P}$-sensitive with reduction set $\mathcal{Q}$. □

Theorem 4.3. *Assume that* $ca_c^* = L_c^\infty$*. Let* $f : L_c^\infty \to (-\infty, \infty]$ *be a quasiconvex (resp. convex), monotone non decreasing (* $X \leq Y$ $\mathcal{P}$*-q.s. implies* $f(X) \leq f(Y)$*) and* $\mathcal{P}$*-sensitive function. The following are equivalent:*

(i) f *is* $\sigma(L_c^\infty, ca_c)$*-lower semi continuous.*

(ii) f *has the Fatou property: for any bounded sequence* $(X_n)_{n \in \mathbb{N}} \subset L_c^\infty$ *converging* $\mathcal{P}$*-q.s. to* $X \in L_c^\infty$ *we have* $f(X) \leq \liminf_{n \to \infty} f(X_n)$*.*

(iii) For any sequence $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ *and* $X \in L_c^\infty$ *such that* $X_n \uparrow X$ $\mathcal{P}$*-q.s. we have that* $f(X_n) \uparrow f(X)$*.*

(iv) f *admits a bidual representation which in the quasiconvex case is*

$$f(X) \quad = \quad \sup_{P \in ca_c \cap \mathcal{M}_1} R\left(E_P[X], P\right), \quad X \in L_c^\infty,$$

with dual function $R : \mathbb{R} \times ca_c \to (-\infty, \infty]$ *given by*

$$R(t, \mu) \quad := \quad \sup_{t' < t} \inf_{Y \in L_c^\infty} \left\{ f(Y) \mid \int Y \, d\mu = t' \right\};$$

and in the convex case the dual representation is

$$f(X) \quad = \quad \sup_{\mu \in (ca_c)_+} \left\{ \int X \, d\mu - f^*(\mu) \right\}, \quad X \in L_c^\infty,$$

where the dual function $f^* : ca_c \to (-\infty, \infty])$ is given by

$$f^*(\mu) \quad := \quad \sup_{Y \in L_c^\infty} \left\{ \int Y \, d\mu - f(Y) \right\}.$$

In addition, if $f(X + c) = f(X) + c$ for every $X \in L_c^\infty$ and $c \in \mathbb{R}$ then f is necessarily convex and

$$f(X) = \sup_{P \in ca_c \cap \mathcal{M}_1} \{ E_P[X] - f^*(P) \}, \quad X \in L_c^\infty.$$

Proof. According to Theorem 3.9 (i) holds if and only if (ii) is satisfied.
(ii) $\Rightarrow$ (iii) is due to

$$f(X) \leq \liminf_{n \to \infty} f(X_n) \leq f(X)$$

where the last ineqaulity follows from monotonicity. Conversely (iii) $\Rightarrow$ (ii) follows by considering $Y_n := \operatorname{ess\,inf}_{k \geq n} X_k$ and noting that $Y_n \uparrow X$ $\mathcal{P}$-q.s. and $f(Y_n) \leq f(X_n)$; see also [FS04, Lemma 4.16].

In the convex case $(i) \Leftrightarrow (iv)$ is Fenchel's Theorem (see [ET99, Proposition 4.1]) together with monotonicity (see [FR02, Corollary 7]).

In the quasiconvex case showing $(i) \Rightarrow (iv)$ is a consequence of the Penot-Volle duality Theorem (see [FM11, Theorem 1.1]) and together with monotonicity (see [C3M09, Lemma 8]), and $(iv) \Rightarrow (iii)$ follows from the monotone convergence theorem and the definition of R. □

4.2 Remarks on the First Fundamental Theorem of Asset Pricing

Pricing theory in mathematical finance is based on the Fundamental Theorem of Asset Pricing, which roughly asserts that in a market without arbitrage opportunities (the so-called no-arbitrage condition) discounted prices are expectations under some risk-neutral probability measure. This characterization is essential to develop a pricing theory for financial instruments which are not traded in the market. In the classical dominated framework on some probability space $(\Omega, \mathcal{F}, P)$ the risk-neutral probability measures are martingale measures for the discounted price process which are equivalent to the reference probability P, see [DS06] for a detailed review.

While it is well-known that the Fundamental Theorem of Asset Pricing in a classical dominated framework is highly related to duality arguments, recent robust approaches to the Fundamental Theorem of Asset Pricing do not use these kind of arguments given the difficulties we outlined in this paper, see e.g. [BN15]. However, under the conditions that we have derived in Section 3 we will see that it is possible to reconcile the Fundamental Theorem of Asset Pricing, the Superhedging Duality, and duality theory on the pair (L_c^∞, ca_c) using the well-known arguments.

Throughout this section we assume that $ca_c^* = L_c^\infty$ holds true. We consider a discrete time market model with terminal time horizont $T \in \mathbb{N}$, and trading times $I := \{0, ..., T\}$. The price process is given by a $\mathcal{P}$-q.s. bounded $\mathbb{R}^d$-valued stochastic process $S = (S_t)_{t\in I} = (S_t^j)_{t\in I}^{j=1,\dots,d}$ on $(\Omega, \mathcal{F})$, and we also assume the existence of a numeraire asset $S_t^0 = 1$ for all $t \in I$. Moreover, we fix a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in I}$ such that the process S is $\mathbb{F}$-adapted. Denote by $\mathcal{H}$ the class of $\mathcal{P}$-q.s. bounded $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable stochastic processes, which is the class of all admissible trading strategies. Let

$$\mathcal{C} := \{X \in L_c^\infty \mid X \le (H \bullet S)_T \ \mathcal{P}\text{-q.s. for some } H \in \mathcal{H}\}$$

where

$$(H \bullet S)_t := \sum_{k=1}^{t} \sum_{j=1}^{d} H_k^j (S_k^j - S_{k-1}^j)$$

is the payoff of the self-financing trading strategy at time $t \in I\backslash\{0\}$ with initial investment $(H \bullet S)_0 = 0$ given by the predictable process $H = (H_t)_{t\in I\backslash\{0\}}$. In this framework the no-arbitrage condition (NA($\mathcal{P}$)) was introduced by [BN15] as given by the following definition.

Definition 4.4. *The described market model is called arbitrage-free, if it satisfies the no-arbitrage condition*

NA($\mathcal{P}$) $(H \bullet S)_T \ge 0$ *$\mathcal{P}$-q.s. implies* $(H \bullet S)_T = 0$ *$\mathcal{P}$-q.s..*

Note that NA($\mathcal{P}$) is equivalent to $\mathcal{C} \cap (L_c^\infty)_+ = \{0\}$.

Lemma 4.5. *Under NA($\mathcal{P}$) if $\mathcal{C}$ is $\mathcal{P}$-sensitive then $\mathcal{C}$ is $\sigma(L_c^\infty, ca_c)$-closed.*

Proof. [BN15, Theorem 2.2] shows that under $NA(\mathcal{P})$ the cone $\mathcal{C}$ is closed under $\mathcal{P}$-q.s. convergence of sequences and therefore $\mathcal{C}$ satisfies (FC). We remark that [BN15, Theorem 2.2] holds in full generality without the product structure on the underlying probability space assumed in [BN15]. Therefore applying Theorem 3.9 we deduce that $\mathcal{C}$ is $\sigma(L_c^\infty, ca_c)$-closed. □

Suppose that $\mathcal{C}$ is $\mathcal{P}$-sensitive. As $\mathcal{C}$ is a $\sigma(L_c^\infty, ca_c)$-closed convex cone, the bipolar Theorem yields

$$\begin{aligned} \mathcal{C} \quad &= \quad \mathcal{C}^{00} = \left\{Y \in L_c^\infty \mid \forall Q \in \mathcal{C}_1^0 : \, E_Q[Y] \le 0\right\} \qquad (4.5) \\ \text{where } \mathcal{C}_1^0 \quad &:= \quad \mathcal{C}^0 \cap \mathcal{M}_1 = \left\{\mu \in \mathcal{C}^0 \mid \mu(1_\Omega) = 1\right\} \\ \text{and } \mathcal{C}^0 \quad &:= \quad \left\{\mu \in ca_c \mid \forall X \in \mathcal{C} : \int X \, d\mu \le 0\right\}. \end{aligned}$$

Notice that since $\mathcal{C} \supseteq -L_+^\infty$ then $\mu \in (ca_c)_+$ for every $\mu \in \mathcal{C}^0$ which explains $\mathcal{C}_1^0$.

Lemma 4.6. $\mathcal{C}_1^0$ *is the set of all martingale measures dominated by the capacity c, that is*

$$\mathcal{C}_1^0 = \{Q \ll \mathcal{P} \mid S \text{ is a } Q\text{-martingale}\}$$

Proof. The proof is well-known and straightforward, so we just give the basic arguments: indeed choose any $Q \in \{Q \ll \mathcal{P} \mid S \text{ is a } Q\text{-martingale}\}$, and let $X \in \mathcal{C}$ and $H \in \mathcal{H}$ such that $X \leq (H \bullet S)_T$ $\mathcal{P}$-q.s. Then $E_Q[X] \leq E_Q[(H \bullet S)_T] = (H \bullet S)_0 = 0$ since $((H \bullet S)_t)_{t \in I}$ is a Q-martingale. Thus $Q \in \mathcal{C}_1^0$.

If $Q \in \mathcal{C}_1^0$ then $E_Q[(H \bullet S)_T] = 0$ for any $H \in \mathcal{H}$ and by choosing appropriate strategies in $\mathcal{H}$ such as $H_t^j = 1_A$ for $A \in \mathcal{F}_{t-1}$, $H_t^i = 0$ for $i \neq j$ and $H_s = 0$ for $s \neq t$ one verifies that Q is a martingale measure for S. □

Theorem 4.7 (First Fundamental Theorem of Asset Pricing)**.**
Suppose $\mathcal{C}$ is $\mathcal{P}$-sensitive. The following are equivalent:

(i) $NA(\mathcal{P})$

(ii) $\mathcal{C}_1^0 \approx \mathcal{P}$

Moreover, the Superhedging Duality holds, that is for any $X \in L_c^\infty$ the minimal superhedging price

$$\pi(X) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \bullet S)_T \geq X \, \mathcal{P}\text{-q.s.}\}$$

satisfies

$$\pi(X) = \sup_{Q \in \mathcal{C}_1^0} E_Q[X]. \tag{4.6}$$

Proof. (i) ⇒ (ii): Clearly, $c(A) = 0$ implies $\sup_{Q \in \mathcal{C}_1^0} Q(A) = 0$ as $\mathcal{C}_1^0 \subset ca_c$. Let $B \in \mathcal{F}$ such that $Q(B) = 0$ for all $Q \in \mathcal{C}_1^0$. Thus $1_B \in \mathcal{C}$ by (4.5), so $1_B = 0$ in L_c^∞ by $NA(\mathcal{P})$, i.e. $c(B) = 0$.

(ii) ⇒ (i): let $H \in \mathcal{H}$ such that $(H \bullet S)_T \geq 0$ $\mathcal{P}$-q.s. Then $Q\{(H \bullet S)_T \geq 0\} = 0$ for every $Q \in \mathcal{C}_1^0$, because $(H \bullet S)_t$ is a Q-martingale with expectation 0, and therefore $(H \bullet S)_T = 0$ $\mathcal{P}$-q.s.

As for the Superhedging Duality note that clearly $\pi(X) \leq \|X\|_{c,\infty}$ since $0 \in \mathcal{H}$, and as $\mathcal{C}_1^0 \neq \emptyset$ ($\mathcal{C} \neq L_c^\infty$) it follows that $\pi(X) > -\infty$. Moreover, by (4.5) we have for any $y \in \mathbb{R}$ that $X - y \in \mathcal{C}$ if and only if $0 \geq \sup_{Q \in \mathcal{C}_1^0} E_Q[X - y] = -y + \sup_{Q \in \mathcal{C}_1^0} E_Q[X]$ which proves (4.6). □

Finally, based on our observations so far we will consider the following stronger no-arbitrage condition. To this end let

$$\mathcal{P}^c = \{P \in \mathcal{M}_1 \mid P \ll c\} \quad \text{and} \quad \widetilde{\mathcal{C}} = \bigcap_{Q \in \mathcal{P}^c} j_Q^{-1} \circ j_Q(\mathcal{C})$$

where $j_Q : (L_c^\infty, \sigma(L_c^\infty, ca_c)) \to (L_Q^\infty, \sigma(L_Q^\infty, L_Q^1))$ is the continuous mapping introduced in Section 3.1. The market fullfils strong no-arbitrage if

$$\widetilde{\mathcal{C}} \cap (L_c^\infty)_+ = \{0\}.$$

Notice that $\widetilde{\mathcal{C}}$ is the smallest $\mathcal{P}$-sensitive set which contains $\mathcal{C}$. We will prove the following:

Theorem 4.8 (Weak Fundamental Theorem of Asset Pricing)**.** *Equivalent are:*

(i) $\widetilde{\mathcal{C}} \cap (L_c^\infty)_+ = \{0\}$

(ii) *there exists a family $\mathcal{Q}$ of martingale measures such that $\mathcal{Q} \approx \mathcal{P}$.*

Before we can prove Theorem 4.8 we need the following result:

Lemma 4.9. *Under $NA(\mathcal{P})$ the set $\widetilde{\mathcal{C}}$ is $\sigma(L_c^\infty, ca_c)$-closed.*

Proof. From $NA(\mathcal{P})$ and [BN15, Theorem 2.2] we know that $\mathcal{C}$ satisfies (FC). Let $C_K := \mathcal{C} \cap \{X \in L_c^\infty \mid \|X\|_{c,\infty} \leq K\}$ for $K > 0$. Repeating the argument given in the proof of Theorem 3.9 we observe that $j_Q^{-1} \circ j_Q(C_K)$ is $\sigma(L_c^\infty, ca_c)$-closed, and thus also

$$A_{Q,K} := j_Q^{-1} \circ j_Q(C_K) \cap \{X \in L_c^\infty \mid \|X\|_{c,\infty} \leq K\} = j_Q^{-1} \circ j_Q(\mathcal{C}) \cap \{X \in L_c^\infty \mid \|X\|_{c,\infty} \leq K\}$$

is $\sigma(L_c^\infty, ca_c)$-closed. Hence, finally also

$$\bigcap_{Q \in \mathcal{P}^c} A_{Q,K} = \widetilde{\mathcal{C}} \cap \{X \in L_c^\infty \mid \|X\|_{c,\infty} \leq K\}$$

is $\sigma(L_c^\infty, ca_c)$-closed. As this holds for every $K > 0$, the Krein-Smulian Theorem implies that $\widetilde{\mathcal{C}}$ is $\sigma(L_c^\infty, ca_c)$-closed. □

Proof of Theorem 4.8. Note that both (i) and (ii) imply $NA(\mathcal{P})$. Thus $\widetilde{\mathcal{C}}$ is $\sigma(L_c^\infty, ca_c)$-closed by Lemma 4.9. As a consequence of the Bipolar Theorem we have

$$\widetilde{\mathcal{C}} = \left\{ X \in L_c^\infty \mid \forall \mu \in \widetilde{\mathcal{C}}_1^0 : \int X \, d\mu \leq 0 \right\}.$$

$$\text{where} \quad \widetilde{\mathcal{C}}_1^0 := \left\{ \mu \in ca_c \mid \mu(1_\Omega) = 1, \forall X \in \widetilde{\mathcal{C}} : \int X \, d\mu \leq 0 \right\} \subset \mathcal{C}_1^0.$$

Let $\mathcal{Q} = \widetilde{\mathcal{C}}_1^0$. (i) $\Leftrightarrow$ (ii) now follows exactly as in the proof of Theorem 4.7. □

A Auxiliary results for Theorem 3.1

Recall the set $\mathcal{Z}$ defined in (3.2).

Proposition A.1. *If $\mathcal{Z} = \emptyset$, then there exists a countable subset $\widetilde{\mathcal{P}} \subset \mathcal{P}$ such that $\widetilde{\mathcal{P}} \approx \mathcal{P}$. The latter implies that there is a probability measure $Q \in \mathcal{M}_1$ such that $\{Q\} \approx \mathcal{P}$.*

Proof. We claim that for each $\varepsilon > 0$, there exists $P_1, \dots, P_n \in \mathcal{P}$ and $\delta > 0$ such that $P_i(A) < \delta$ for all $i = 1, \dots, n$ implies that for all $P \in \mathcal{P}$ we have $P(A) < \varepsilon$. Suppose this is not the case. Then there exists $\varepsilon > 0$ such that for any $P_1 \in \mathcal{P}$ there is $A_1 \in \mathcal{F}$ and $P_2 \in \mathcal{P}$ satisfying

$$P_1(A_1) < 1/2 \quad \text{and} \quad P_2(A_1) \geq \varepsilon.$$

Then there also exists $A_2 \in \mathcal{F}$ and $P_3 \in \mathcal{P}$ such that

$$P_1(A_2) < 1/4,\ P_2(A_2) < 1/4 \quad \text{while} \quad P_3(A_2) \geq \varepsilon.$$

Continuing this procedure we find sequences $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ and $(P_n)_{n\in\mathbb{N}} \in \mathcal{P}$ such that

$$P_i(A_n) < \frac{1}{2^n},\ i = 1, \dots, n, \quad \text{and} \quad P_{n+1}(A_n) \geq \varepsilon.$$

Consider $N := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$. Then $P_i(N) = 0$ for each $i \in \mathbb{N}$, because for all $n > (i-1)$

$$P_i(N) \leq \sum_{k=n}^{\infty} P_i(A_k) \leq \frac{1}{2^{n-1}}.$$

Hence, replacing the above sequence A_n by $B_n := A_n \setminus N$, $n \in \mathbb{N}$, we still have

$$P_i(B_n) < \frac{1}{2^n},\ i = 1, \dots, n, \quad \text{and} \quad P_{n+1}(B_n) \geq \varepsilon.$$

Now let $E_n := \bigcup_{k\geq n} B_k$, $n \in \mathbb{N}$. It follows that $E_n \downarrow \emptyset$. However, for each $n \in \mathbb{N}$

$$c(E_n) \geq P_{n+1}(E_n) \geq P_{n+1}(B_n) \geq \varepsilon$$

which contradicts $\mathcal{Z} = \emptyset$.

Now let $\delta_n > 0$ and let $P_1^{(n)}, \dots, P_{m(n)}^{(n)} \in \mathcal{P}$ be such that for all $P \in \mathcal{P}$ it holds $P(A) < 1/n$ whenever $P_i^{(n)}(A) < \delta_n$ for all $i = 1, \dots, m(n)$. Define

$$\mu := \sum_{n=1}^{\infty} \sum_{i=1}^{m(n)} \frac{1}{2^n} \frac{1}{2^i} P_i^{(n)}.$$

Then $\mu \in ca_+$, and $\mu(A) = 0$ implies that $P_i^{(n)}(A) = 0$ for all $i = 1, \dots, m(n)$ and $n \in \mathbb{N}$. Eventually this implies that for all $P \in \mathcal{P}$ we have $P(A) < 1/n$ for all $n \in \mathbb{N}$, hence $P(A) = 0$. Thus

$$\widetilde{\mathcal{P}} := \{P_i^{(n)} \mid i \in \{1, \dots, m(n)\}, n \in \mathbb{N}\} \quad \text{and} \quad Q := \frac{1}{\mu(\Omega)}\mu$$

satisfy the assertion. $\square$

Proposition A.2. *Let $(B, \|\cdot\|)$ be a Banach lattice of (equivalence classes of) random variables on $(\Omega, \mathcal{F})$ containing all simple random variables such that the order $\leq$ on B satisfies $0 \leq 1_A \leq 1_{A'}$ whenever $A \subset A'$ for $A, A' \in \mathcal{F}$. If $B^* \subset ca$, in the sense that every $l \in B^*$ is of type*

$$l(X) = \int X \, d\mu, \quad X \in B,$$

for some $\mu \in ca$, then $\|1_{A_n}\| \to 0$ $(n \to \infty)$ for all $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$. Conversely, if $\|1_{A_n}\| \to 0$ $(n \to \infty)$ for all $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$, then for every $l \in B^*$ there is a $\mu \in ca$ such that $l(Y) = \int Y \, d\mu$ for all simple random variables Y.

Proof. Suppose that $B^* \subset ca$ and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$. Then $1_{A_n} \to 0$ with respect to $\sigma(B, B^*)$ since every element in B^* corresponds to a σ-additive measure. Hence,

$$0 \in \overline{co\{1_{A_n} \mid n \in \mathbb{N}\}}$$

where the closure is taken in the $\sigma(B, B^*)$-topology. As the closed convex set in the $\sigma(B, B^*)$-topology and in the norm topology coincide, we have that there is a sequence of convex combinations

$$c_k := \sum_{i=1}^{m(k)} a_i(k) 1_{A_{n_i(k)}}, \quad k \in \mathbb{N},$$

where $a_i(k) \in \mathbb{R}$ and $n_1(k) \leq n_2(k) \leq \ldots \leq n_{m(k)}(k)$ for all $k \in \mathbb{N}$ such that $\|c_k\| \to 0$ for $k \to \infty$. Moreover, since $0 \in \overline{co\{1_{A_n} \mid n \geq N\}}$ for any $N \in \mathbb{N}$, we may assume that $n_1(k) \leq n_1(k+1)$ for all $k \in \mathbb{N}$. However, $c_k \geq 1_{A_k}$ where $A_k = A_{n_{m(k)}(k)}$, because $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Thus, as $\|\cdot\|$ is a lattice norm, the subsequence 1_{A_k} converges to 0 in norm and hence also 1_{A_n} converges to 0 in the norm topology (again due to $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$).

Finally suppose that $\|1_{A_n}\| \to 0$ $(n \to \infty)$ for all $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $A_n \downarrow \emptyset$. Then for any $l \in B^*$, the set function

$$\mu(A) := l(1_A), \quad A \in \mathcal{F},$$

is σ-additive. By linearity of l we deduce that $l(X) = \int X \, d\mu$ for all simple random variables X. □

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