COHERENT FOREIGN EXCHANGE MARKET MODELS

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Abstract. A model describing the dynamics of a foreign exchange (FX) rate should preserve the same level of analytical tractability when the inverted FX process is considered. We show that affine stochastic volatility models satisfy such a requirement. Such a finding allows us to use affine stochastic volatility models as a building block for FX dynamics which are functionally-invariant with respect to the construction of suitable products/ratios of rates thus generalizing the model of [12].

JEL Classification G12, G15.

1. Introduction

The foreign exchange (FX) market is the largest and most liquid financial market in the world. According to the Bank of International settlements (see [36]), the daily global FX market volume (or turnover) in 2010 was about 3981 billion dollars. This huge amounts breaks-down into spot transactions (1490 billion), FX swaps (1765 billion) and FX options (207 billion). These figures give an idea of the relevance of the market for FX products and FX options in particular and hence provide a reasonable grounding for questions focusing on it, see also [8]. The increasing popularity of complex insurance products featuring FX risk highlights the relevance of the topic also for the insurance industry, see [40].

When we look at the market for FX options we observe phenomena which may be summarized in two main categories.

- Stylized facts regarding the underlying securities.
- Stylized facts regarding the FX implied volatility.

As far as the first category is concerned, we have that, unlike e.g. in the equity market, we may consider both the underlying and its inverse. To be more specific, if S denotes the EURUSD exchange rate (which is the price in dollars of 1 euro), the reciprocal of S, $1/S$, denotes then the USDEUR exchange rate, i.e. the price in euros of 1 dollar. More generally, this kind of reasoning may be further extended as we will see, and hence suitable ratios/products of exchange rates are still exchange rates. This key feature of exchange rates has an implication on the set of requirements that a realistic model should satisfy. In particular, in the simple two-economy case, assuming a certain stochastic dynamics for the exchange rate S has been postulated, it is not a priori clear if the process for the inverted exchange rate process $1/S$ shares the same level of analytical tractability of the original process S.

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Concerning the second problem, when we look at implied volatility of FX options we observe different values for different levels of moneyness/maturity, which are summarized in the so-called volatility surface.

Extending our views to multiple currencies, and hence by looking at a a variance covariance matrix of currencies, we observe that both variances and covariances are stochastic, pointing out the need for a model in which not only we have stochastic volatility on single exchange rates, but also stochastic correlations among them. An example of this phenomenon is visualized in Figure 1, where we perform a very simple estimation on rolling windows of the variance covariance matrix of two liquid exchange rates $EURUSD$ and $EURJPY$.

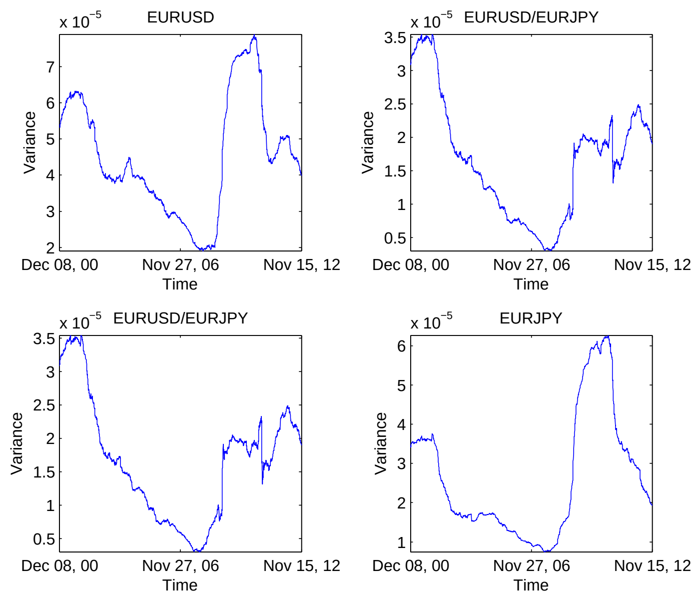

FIGURE 1. Time series of variance and covariances for EURUSD and EURJPY. Estimation performed using a rolling sample of 500 data points on time series with daily frequency.

Combining the two cathegories of stylized facts above into a single model which is at the sime time coherent under the inversion of the currency, while able to capture the features of the smile, is a non-trivial task. Ever since the contribution of [21], which represents an adaptation of the Black-Scholes model [5], it has become quite common to take a model initially imagined for e.g. stock options, and employ it, with minor adjustments, for the evaluation of FX options. An example in this sense is the model of [26], whose FX adaptation is discussed in many references, e.g. [10], [28]. Other models which are employed in an FX setting are e.g. [39] or [27], see the account in [33]. Our aim in the present article is to identify a class of stochastic processes for the evolution of exchange rates which

- allows for the realistic description of stochastic volatility/correlation effects and discountinuous paths;
- is closed under inversion or, more generally, under suitable products/ratios of processes.

This paper is outlined as follows: in Section 2, we introduce our main assumptions. In section 3 we present our main result: we consider the class of affine stochastic volatility models, as introduced by [30] and show that the process for the inverted exchange rate is still an affine process which is as analytically tractable as the starting model. Given this result, we can then look at more advanced situations and then consider triangles or more general geometric structures of FX rates. The idea of Section 4 is to use the previous findings in the two economy case, in order to present an example of model which is functionally symmetric under suitable product/ratios among exchange rates. The model is an extension of the multifactor stochastic volatility model introduced by [12].

2. The setting

2.1. **Basic traded assets and coherent models.** We specify a general market setting consisting of two economies. We assume the existence of a risk-free money market account for each currency area. We denote by $B^d(t), B^f(t)$ the domestic and the foreign money market accounts respectively, which are solutions to the following ODEs

$$dB^i(t) = r_i dt,\ B^i(0) = 1,\ i = d, f, \tag{2.1}$$

where we assume, for the sake of simplicity, that $r_i,\ i = d, f$ are real valued constants.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}_i)\ i = d, f$ be a filtered probability space, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions. We also let $\mathcal{F}_0$ be the trivial sigma algebra. Let us postpone for the moment the treatment of the family of probability measures $\mathbb{Q}_i,\ i = d, f$. On this probability space we will be considering in general two stochastic processes: $S = (S(t))_{t\geq 0}$ and $S^{-1} = \left(S(t)^{-1}\right)_{t\geq 0}$. S will be employed to model the foreign exchange rate in the usual FORDOM convention i.e., if S is a model for EURUSD and $S = 1.30$, then we say that one Euro is worth 1.30 dollars. In a dual way, we let S^{-1} be a model for the USDEUR exchange rate, thus capturing the point of view of a European investor. Given the processes defined above, agents from the two economies may trade the following assets.

- The domestic agent may trade
  1. In the domestic money market account $B^d = \left(B^d(t)\right)_{t\geq 0}$;
  2. In the foreign money market account $\tilde{B}^f = \left(B^f(t)S(t)\right)_{t\geq 0}$.
- The foreign agent may trade
  1. In the foreign money market account $B^f = \left(B^f(t)\right)_{t\geq 0}$;
  2. In the domestic money market account $\tilde{B}^d = \left(B^d(t)S(t)^{-1}\right)_{t\geq 0}$.

We now concern ourselves with the viability of the market setting above. To this end we introduce the following notation: whenever they exist, we denote via

- $\mathbb{Q}_d$ the probability measure such that

$$\mathbb{E}^{\mathbb{Q}_d}\left[\left.\frac{\tilde{B}^f(T)}{B^d(T)}\right|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_d}\left[\left.\frac{B^f(T)S(T)}{B^d(T)}\right|\mathcal{F}_t\right] = \frac{B^f(t)S(t)}{B^d(t)},$$

and call it domestic risk neutral measure;
- $\mathbb{Q}_f$ the probability measure such that

$$\mathbb{E}^{\mathbb{Q}_f}\left[\frac{\tilde{B}^d(T)}{B^f(T)}\middle|\mathcal{F}_t\right]=\mathbb{E}^{\mathbb{Q}_f}\left[\frac{B^d(T)S(T)^{-1}}{B^f(T)}\middle|\mathcal{F}_t\right]=\frac{B^d(t)S(t)^{-1}}{B^f(t)},$$

and call it foreign risk neutral measure.

We now introduce the following assumption.

Assumption 1. *We assume that a $\mathbb{Q}_d$-risk neutral measure exists, i.e. we assume that the process*

$$Z=(Z(t))_{t\geq 0}:=\left(\frac{B^f(t)S(t)}{S(0)B^d(t)}\right)_{t\geq 0}$$

is a true $\mathbb{Q}_d$-martingale with $Z(0)=1$.

Under this assumption, we have that the market model we are considering is free of arbitrage, see Definiton 9.2.8 and Theorem 14.1.1 in [14]. In general, we will not assume that $\mathbb{Q}_d$ is unique, as we will be concerned with stochastic volatility models possibly featuring jumps, which provide typical examples of incomplete markets. In such a setting the particular measure $\mathbb{Q}_d$ will be determined as the result of a calibration to market data.

A direct consequence of Assumption 1, is that the process Z may be employed so as to define the density process of the risk neutral measure $\mathbb{Q}_f$. More explicitly, we have

$$\left.\frac{\partial\mathbb{Q}_f}{\partial\mathbb{Q}_d}\right|_{\mathcal{F}_t}=\frac{S(t)B^f(t)}{S(0)B^d(t)}, \tag{2.2}$$

see Theorem 1 in [22]. The process above is the change of measure which is found in the classical literature on FX markets, see e.g. Section 2.9 in [6]. While this is a well established fact, we would like to underline, by means of the following examples, that the change of measure above may introduce significant changes in the model specification under different pricing measures.

Example 1. *Let us consider the GARCH stochastic volatility model, which is studied in depth in* [32]*. The dynamics are given by*

$$\frac{dS(t)}{S(t)}=(r_d-r_f)dt+\sqrt{V(t)}dW^{1,\mathbb{Q}_d}(t), \tag{2.3}$$

$$dV(t)=(\omega-\alpha V(t))dt+V(t)\left(\rho dW^{1,\mathbb{Q}_d}(t)+\sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)\right), \tag{2.4}$$

which is specified under the domestic risk neutral measure. The density process between the foreign and the domestic risk neutral measure is given by

$$\left.\frac{\partial\mathbb{Q}_f}{\partial\mathbb{Q}_d}\right|_{\mathcal{F}_t}=\exp\left\{\int_0^t\sqrt{V(s)}dW^1_s-\frac{1}{2}\int_0^t V(s)ds\right\}. \tag{2.5}$$

Notice that this change of measure, which is typical for stochastic volatility models with non-zero correlations, implies that the second Brownian motion W^2 is not affected by the measure change, so $W^{2,\mathbb{Q}_f}=W^{2,\mathbb{Q}_d}$, see [32, p. 257]*. Under the* *foreign risk neutral measure the dynamics of the inverse exchange rate are now given by (compare with* [32, Eq. 3.6 p. 257]*)*

$$\frac{dS^{-1}(t)}{S^{-1}(t)} = (r_f - r_d)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_f}(t), \tag{2.6}$$

$$\begin{aligned} dV(t) &= (\omega - \alpha V(t) + \rho V(t)^{3/2})dt \\ &\quad + V(t)\left(-\rho dW^{1,\mathbb{Q}_f}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_f}(t)\right), \end{aligned} \tag{2.7}$$

which is not a GARCH stochastic volatility model.

Example 2. *Let us consider the Hull-White stochastic volatility model*

$$\frac{dS(t)}{S(t)} = (r_d - r_f)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_d}(t), \tag{2.8}$$

$$dV(t) = \mu V(t)dt + \xi V(t)\left(\rho dW^{1,\mathbb{Q}_d}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)\right). \tag{2.9}$$

Also in this case we have a pathological situation: under the foreign risk neutral measure the variance will not follow a geometric Brownian motion as under the starting measure

$$\frac{dS^{-1}(t)}{S^{-1}(t)} = (r_f - r_d)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_f}(t), \tag{2.10}$$

$$\begin{aligned} dV(t) &= \left(\mu V(t) + \xi V(t)^{3/2}\rho\right)dt \\ &\quad + \xi V(t)\left(-\rho dW^{1,\mathbb{Q}_f}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_f}(t)\right). \end{aligned} \tag{2.11}$$

By proceeding along the same lines, it is also possible to show that the SABR stochastic volatility model suffers from the same lack of symmetry. On the contrary, when we consider the stochastic volatility model by Heston, see [26], we observe that the structure of the model is instead preserved, as shown in [13]. More specifically, under the measure $\mathbb{Q}_d$, the dynamics of the instantaneous variance factor of S are those of a square root process

$$\begin{aligned} dV(t) &= \kappa(\theta - V(t))dt \\ &\quad + \sigma\sqrt{V(t)}\left(\rho dW^{1,\mathbb{Q}_d}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)\right), \end{aligned}$$

for $\sigma > 0, \kappa \in \mathbb{R}$, s.t. $\theta\kappa > 0$ whereas, under the measure $\mathbb{Q}_f$, the instantaneous variance of the inverted exchange rate is still a square root process where the parameters under $\mathbb{Q}_f$ are modified as follows

$$\kappa^{\mathbb{Q}_f} = \kappa - \sigma\rho \tag{2.12}$$

$$\theta^{\mathbb{Q}_f} = \frac{\kappa\theta}{\kappa^{\mathbb{Q}_f}} \tag{2.13}$$

$$\rho^{\mathbb{Q}_f} = -\rho. \tag{2.14}$$

In view of the examples above we may say that an FX model is **coherent** if the processes for S and $1/S$ belong to the same class.

2.2. **The foreign-domestic parity.** We can further extend the market setting by introducing European options written both on the exchange rate and its inverse. More specifically

Assumption 2. *The domestic and the foreign agent may create positions in the following assets*

- *European call/put options written on S*
- *European call/put options written on S^{-1}.*

It is well known, see e.g. [7] or [15], that option prices are intimately linked to characteristic functions of log-prices. To this end, let us define, for fixed $t \geq 0$ and all $v \in \mathbb{C}$ such that the expectations exist, the following characteristic functions

$$\varphi_i : \mathbb{C} \to \mathbb{C},\ i = d, f,$$

$$\varphi_d(v) := \mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}v \log S(t)}\right],\ v \in \mathbb{C}, \tag{2.15}$$

$$\varphi_f(v) := \mathbb{E}^{\mathbb{Q}_f}\left[e^{\mathtt{i}v \log S(t)^{-1}}\right],\ v \in \mathbb{C}, \tag{2.16}$$

where $\mathtt{i}$ denotes the imaginary unit. Let us define $F(t) := S_0 e^{(r_d - r_f)t}$. From Assumption 1 we can obtain the following

Proposition 1. *Under Assumption 1 the characteristic functions φ_d, φ_f obey the following relation*

$$\varphi_f(u) = F(t)^{-1}\varphi_d(-u - \mathtt{i}). \tag{2.17}$$

for $u \in \mathbb{R}$.

Proof. Using the definitions of φ_d, φ_f in (2.15), (2.16), coupled with Assumption 1 and the Bayes rule, allows us to write

$$\begin{aligned}
\varphi_f(u) &= \mathbb{E}^{\mathbb{Q}_f}\left[e^{\mathtt{i}u \log S(t)^{-1}}\right] \\
&= \frac{1}{Z(0)}\mathbb{E}^{\mathbb{Q}_d}\left[Z(t)e^{\mathtt{i}u \log S(t)^{-1}}\right] \\
&= \frac{B^d(0)B^f(t)}{S(0)B^f(0)B^d(t)}\mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}(-u-\mathtt{i})\log S(t)}\right] \\
&= F(t)^{-1}\varphi_d(-u - \mathtt{i}).
\end{aligned}$$

The finiteness of $\varphi_d(-u-\mathtt{i})$ is guaranteed by the martingale property of the density process Z, hence the proof is complete. $\square$

When the basic market model is enriched with the above derivative securities, we can investigate the foreign domestic parity, which is a no-arbitrage relationship, which links call options written on an FX rate to put options on the inverse FX rate. By foreign domestic parity we mean the following relation

$$\mathcal{CALL}\left(S_0, K, r_d, r_f, T\right) = S_0 K \mathcal{PUT}\left(\frac{1}{S_0}, \frac{1}{K}, r_f, r_d, T\right). \tag{2.18}$$

To get an understanding of the relation we follow [41] and take as an example a call on EURUSD (recall that EURUSD is quoted in FORDOM terms, meaning that we are looking at the dollar value of one euro and so we take the perspective of an American investor). The payoff $(S(T) - K)^+$ is worth $\mathcal{CALL}\left(S_0, K, r_d, r_f, T\right)$ for the American investor, hence $\mathcal{CALL}\left(S_0, K, r_d, r_f, T\right)/S_0$ euros. This $EUR-$call may be viewed as the payoff $K\left(K^{-1}-S(T)^{-1}\right)^{+}$. This payoff for a European investor is worth $K\mathcal{PUT}\left(\frac{1}{S_0},\frac{1}{K},r_f,r_d,T\right)$. Absence of arbitrage tells us that the two values must agree. As a Corollary to Proposition 1, we can state a corrected proof of the following result, originally proved by [13].

Corollary 1. *Under Assumption 1 then the foreign domestic parity holds.*

Proof. See Appendix. □

In the following, we return to our main object of investigation, namely the identification of a wide class of models which are coherent.

3. Affine stochastic volatility models

We consider affine stochastic volatility models, in the sense of [29], [30]. More precisely, we consider an asset price $S=(S(t))_{t\geq 0}$ of the form

$$S(t)=\exp\left\{(r_d-r_f)\,t+X(t)\right\}\ \ t\geq 0, \tag{3.1}$$

so that $X=(X(t))_{t\geq 0}$ is a model for the discounted stock log-price process. We let $V=(V(t))_{t\geq 0}$ be another process, with $V_0>0$ a.s., which may represent the instantaneous variance of $(X(t))_{t\geq 0}$ or may control the arrival rate of its jumps. We assume that the joint process $(X,V)=(X(t),V(t))_{t\geq 0}$ satisfies the assumptions **A1, A2, A3, A4** in [30] and call it affine stochastic volatility model. More precisely we assume that

- **A1** (X,V) is a stochastically continuous, time-homogeneous Markov process with state space $\mathbb{R}\times\mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}:=[0,\infty)$,
- **A2** The cumulant generating function of $(X(t),V(t))$ is of a particular affine form: there exist functions $\phi(t,v,w)$ and $\psi(t,v,w)$ such that

$$\log\mathbb{E}^{\mathbb{Q}_d}\left[\exp\left(vX(t)+wV(t)\right)\right]=\phi(t,v,w)+\psi(t,v,w)V_0+vX_0$$

for all $(t,v,w)\in\mathcal{U}$ for $\mathcal{U}$ defined as

$$\begin{aligned}\mathcal{U}:=\Big\{(t,v,w)\in\mathbb{R}_{\geq 0}\times\mathbb{C}^2\Big|\ &\mathbb{E}^{\mathbb{Q}_d}\left[\left|\exp\left(vX(t)+wV(t)\right)\right|\right]\\ &=\mathbb{E}^{\mathbb{Q}_d}\left[\exp\left(\Re(v)X(t)+\Re(w)V(t)\right)\right]<\infty\Big\}.\end{aligned}$$

Assumptions **A1, A2** make the process $(X(t),V(t))_{t\geq 0}$ affine in the sense of Definition 2.1 in [18]. For our purposes, this implies that the characteristic function of the logarithmic exchange rate is of a particularly nice form

$$\varphi_d(u)=\mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathrm{i}u\log S(t)}\right]=e^{\mathrm{i}u(r_d-r_f)t+\phi(t,\mathrm{i}u,0)+V_0\psi(t,\mathrm{i}u,0)+X_0\mathrm{i}u},$$

for $u\in\mathbb{R}$. The functions ϕ,ψ may be easily characterized. In fact, from Theorem 2.1 in [30] we know that ϕ,ψ satisfy, for $(t,v,w)\in\mathcal{U}$, the generalized Riccati equations

$$\frac{\partial}{\partial t}\phi(t,v,w)=F(v,\psi(t,v,w)),\ \ \phi(0,v,w)=0, \tag{3.2}$$

$$\frac{\partial}{\partial t}\psi(t,v,w)=R(v,\psi(t,v,w)),\ \ \psi(0,v,w)=w. \tag{3.3}$$

The results presented in [18] imply that the RHS in the system of ODE above, i.e. the functions $F(v,\psi(t,v,w))$, and $R(v,\psi(t,v,w))$ are of Lévy-Khintchine form, i.e.:

$$
\begin{aligned}
F(v,\psi) &= \begin{pmatrix} v & \psi \end{pmatrix} \frac{a}{2} \begin{pmatrix} v \\ \psi \end{pmatrix} + b \begin{pmatrix} v \\ \psi \end{pmatrix} \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - v\frac{x}{1+x^2} \right) m\left(dx, dy\right), \qquad (3.4)\\
R(v,\psi) &= \begin{pmatrix} v & \psi \end{pmatrix} \frac{\alpha}{2} \begin{pmatrix} v \\ \psi \end{pmatrix} + \beta \begin{pmatrix} v \\ \psi \end{pmatrix} \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - v\frac{x}{1+x^2} - \psi\frac{y}{1+y^2} \right) \mu\left(dx, dy\right). \qquad (3.5)
\end{aligned}
$$

Moreover the set of parameters $(a, \alpha, b, \beta, m, \mu)$ satisfy the following admissibility conditions

- a, α are positive semi-definite 2×2 matrices with $a_{12} = a_{21} = a_{22} = 0$,
- $b \in \mathbb{R} \times \mathbb{R}_{\geq 0}$, $\beta \in \mathbb{R}^2$,
- m, μ are Lévy measures on $\mathbb{R} \times \mathbb{R}_{\geq 0}$, such that

$$\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left((x^2 + y) \wedge 1 \right) m(dx, dy) < \infty.$$

To gain an intuition on the role of the parameters we observe that F and R represent respectively the constant and the state-dependent characteristics of the vector process $(X(t), V(t))_{t\geq 0}$. More precisely

- $a + \alpha V(t)$ is the instantaneous covariance matrix,
- $b + \beta V(t)$ is the drift,
- $m + \mu V(t)$ is the Lévy measure.

In the following, we will consider a simplified version of the system of ODE (3.2), (3.3), (3.4) and (3.5). In particular, we will assume that we have at most jumps of finite variation for the positive (variance) component. Moreover, having applications in mind, we parametrize the linear diffusion coefficient α by means of a coefficient $\rho \in [-1, 1]$ and we include, in the jump transform, the term coming from the martingale condition for the asset price. Finally we will assume the following conditions

- $b_1 = -\frac{a_{11}}{2}$,
- $\beta_1 = -\frac{\alpha_{11}}{2}$.

Which correspond to the condition $F(1, 0) = R(1, 0) = 0$ in Theorem 2.5 in [30]. In summary we will be assuming that the process is conservative and is a martingale, and also $R(u, 0) \neq 0$ for some $u \in \mathbb{R}$ which excludes models where the distribution of the log asset price does not depend on the variance. These are the assumptions

A3, A4 in [30]. The functions F, R are then of the following form

$$F(v,\psi) = -u^2\frac{a_{11}}{2} + b\begin{pmatrix} v \\ \psi \end{pmatrix} + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - vx\right) - v\left(e^x - 1 - x\right) m\left(dx, dy\right), \tag{3.6}$$

$$R(v,\psi) = \frac{1}{2}\begin{pmatrix} v & \psi \end{pmatrix}\begin{pmatrix} \alpha_{11} & \rho\sqrt{\alpha_{11}\alpha_{22}} \\ \rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22} \end{pmatrix}\begin{pmatrix} v \\ \psi \end{pmatrix} + \beta\begin{pmatrix} v \\ \psi \end{pmatrix} + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - vx\right) - v\left(e^x - 1 - x\right)\mu\left(dx, dy\right). \tag{3.7}$$

We proceed to state our main result on the coherency of affine stochastic volatility models. For the sake of clarity, let us introduce the notation $\phi^{\mathbb{Q}_i}, \psi^{\mathbb{Q}_i},\ i = d, f$ so as to denote the cumulant generating function under the foreign and the domestic risk neutral measure. Similarly, we also introduce $F^{\mathbb{Q}_i}, R^{\mathbb{Q}_i}\ i = d, f$.

Theorem 1. *Let $(X,V) = (X(t),V(t))_{t\geq 0}$ be an affine stochastic volatility model for the exchange rate process S of the form* (3.1) *whose affine representation is given by* (3.6), (3.7). *Then the foreign risk-neutral martingale measure $\mathbb{Q}_f$ has the following density process with respect to $\mathbb{Q}_d$*

$$\left.\frac{\partial\mathbb{Q}_f}{\partial\mathbb{Q}_d}\right|_{\mathcal{F}_t} = e^{X_t - X_0}, \tag{3.8}$$

and the model for the inverted exchange rate S^{-1} is still an affine stochastic volatility model with

$$\phi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w),$$
$$\psi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w),$$

and corresponding characteristics

$$F^{\mathbb{Q}_f}(\mathtt{i}u, \psi) = F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), \psi), \tag{3.9}$$

$$R^{\mathbb{Q}_f}(\mathtt{i}u, \psi) = R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), \psi), \tag{3.10}$$

for $(t, \mathtt{i}(-u-\mathtt{i}), w) \in \mathcal{U}$ with $u \in \mathbb{R}$.

Proof. The form of the density process (3.8) and the martingale property are immediate given our assumptions. It remains to show that the model for the inverted exchange rate is still an affine stochastic volatility model with characteristics (3.9)-(3.10) and for this part we follow the arguments of [29, Theorem 4.14]. We have that

$$\mathbb{E}^{\mathbb{Q}_f}\left[e^{\mathtt{i}u\log S(t)^{-1}+wV(t)}\right]=\mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}u\log S(t)^{-1}+wV(t)}Z(t)\right]\frac{1}{Z(0)}$$
$$=e^{\mathtt{i}u(r_f-r_d)t}\mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}(-u-\mathtt{i})X(t)+wV(t)}\right]e^{-X(0)}$$
$$=e^{\mathtt{i}u(r_f-r_d)t+\phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)+\psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)V(0)+\mathtt{i}(-u-\mathtt{i})X(0)-X(0)}$$
$$e^{\mathtt{i}u\left(\log S(0)^{-1}+(r_f-r_d)t\right)+\phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)+\psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)V(0)}$$
$$e^{\mathtt{i}u\left(\log S(0)^{-1}+(r_f-r_d)t\right)+\phi^{\mathbb{Q}_f}(\mathtt{i}u,w)+\psi^{\mathbb{Q}_f}(\mathtt{i}u,w)V(0)}$$

where we set $\phi^{\mathbb{Q}_f}(\mathtt{i}u,w)=\phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)$ and $\psi^{\mathbb{Q}_f}(\mathtt{i}u,w)=\psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),w)$. From the system of generalized Riccati ODEs (3.2)-(3.3) we finally obtain (3.9) and (3.10) upon direct inspection. □

Corollary 2. *Under the Assumption of Theorem 1, the admissible parameter sets for S and S^{-1} are related as follows*

(1) $b_1^{\mathbb{Q}_f}=-a_{11}-b_1$,
(2) $\rho^{\mathbb{Q}_f}=-\rho$,
(3) $\beta_1^{\mathbb{Q}_f}=-\alpha_{11}-\beta_1$,
(4) $\beta_2^{\mathbb{Q}_f}=\beta_2+\rho\sqrt{\alpha_{11}\alpha_{22}}$,
(5) $m^{\mathbb{Q}_f}(dx,dy)=e^x m(dx,dy)$,
(6) $\mu^{\mathbb{Q}_f}(dx,dy)=e^x\mu(dx,dy)$.

Proof. From Theorem 1 we know that

$$F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)=F^{\mathbb{Q}_f}(\mathtt{i}u,\psi), \tag{3.11}$$

$$R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)=R^{\mathbb{Q}_f}(\mathtt{i}u,\psi). \tag{3.12}$$

In the steps below, we analyze separately drift, diffusion and jumps coefficient of affine stochastic volatility models.

Step 1: constant diffusion and drift coefficients. We compute $F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the no-jump case

$$F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)=\frac{\mathtt{i}^2(-u-\mathtt{i})^2}{2}a_{11}+\mathtt{i}(-u-\mathtt{i})b_1+\psi b_2$$
$$=\frac{1}{2}\mathtt{i}^2u^2a_{11}+\begin{pmatrix}-a_{11}-b_1 & b_2\end{pmatrix}\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix}+\frac{a_{11}}{2}+b_1.$$

We notice that (3.9) is satisfied if and only if

$$b_1^{\mathbb{Q}_f}=-a_{11}-b_1$$
$$b_1=-\frac{a_{11}}{2}$$

Where the second condition is dictated by the martingale property for the log-exchange rate and is then satisfied by assumption. These conditions can be easily understood by looking at the Black-Scholes model.

Step 2: linear diffusion and drift coefficients. Let us compute, under $\mathbb{Q}_d$, the function $R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the no-jump case.

$$\begin{aligned}
&R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)\\
&=\frac{1}{2}\begin{pmatrix}\mathtt{i}(-u-\mathtt{i}) & \psi\end{pmatrix}\begin{pmatrix}\alpha_{11} & \rho\sqrt{\alpha_{11}\alpha_{22}}\\ \rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\mathtt{i}(-u-\mathtt{i})\\ \psi\end{pmatrix}\\
&+\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}\begin{pmatrix}\mathtt{i}(-u-\mathtt{i})\\ \psi\end{pmatrix}\\
&=\frac{1}{2}\begin{pmatrix}\mathtt{i}u & \psi\end{pmatrix}\begin{pmatrix}\alpha_{11} & -\rho\sqrt{\alpha_{11}\alpha_{22}}\\ -\rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix}\\
&+\begin{pmatrix}-\alpha_{11}-\beta_1 & \beta_2+\rho\sqrt{\alpha_{11}\alpha_{22}}\end{pmatrix}\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix}+\frac{\alpha_{11}}{2}+\beta_1.
\end{aligned}$$

The functional form of the model is preserved if and only if we have

$$\begin{aligned}
\rho^{\mathbb{Q}_f}&=-\rho,\\
\beta_1^{\mathbb{Q}_f}&=-\alpha_{11}-\beta_1,\\
\beta_2^{\mathbb{Q}_f}&=\beta_2+\rho\sqrt{\alpha_{11}\alpha_{22}},\\
\beta_1&=-\frac{\alpha_{11}}{2},
\end{aligned}$$

where again the final condition is dictated by the martingale property for the log-exchange rate and is then satisfied by assumption. These conditions may be easily visualized by considering the Heston [26] model and where first obtained by [13].

Step 3: constant and linear jump coefficients. We concentrate on the linear part of the cumulant generating function, since the precedure is completely analogous for the constant part. We compute $R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the pure-jump case.

$$\begin{aligned}
R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)&=\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}}\left(e^{\mathtt{i}(-u-\mathtt{i})x+\psi y}-1-\mathtt{i}(-u-\mathtt{i})x\right)\\
&-\mathtt{i}(-u-\mathtt{i})\left(e^{x}-1-x\right)\mu\left(dx,dy\right)\\
&=\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}}\left(e^{-\mathtt{i}ux+\psi y}-1-\mathtt{i}ue^{-x}+\mathtt{i}u\right)e^{x}\mu(dx,dy).
\end{aligned}$$

We look then at the jump transform of the inverted exchange rate under the foreign risk neutral measure and obtain

$$\begin{aligned}
R^{\mathbb{Q}_f}(\mathtt{i}u,\psi)&=\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}}\left(e^{-\mathtt{i}ux+\psi y}-1+\mathtt{i}ux\right)\\
&-\mathtt{i}u\left(e^{-x}-1+x\right)\mu^{\mathbb{Q}_f}\left(dx,dy\right)\\
&=\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}}\left(e^{-\mathtt{i}ux+\psi y}-1-\mathtt{i}ue^{-x}+\mathtt{i}u\right)\mu^{\mathbb{Q}_f}(dx,dy),
\end{aligned}$$

from which we obtain $\mu^{\mathbb{Q}_f}(dx,dy)=e^{x}\mu(dx,dy)$. The result on the constant jump coefficient is completely analogous and this completes the proof. □

Remark 1. *The results of Therem 1 and Corollary 2 can be directly extended along the following directions, that we omit for the sake of brevity.*

- *It is possible to choose a multivariate volatility factor process $V=(V(t))_{t\geq 0}$ taking values in $\mathbb{R}_+^d$ or even a matrix variate volatility driver taking values* *on the cone of positive semidefinite $d \times d$ matrices (See* [23] *for an example based on the Wishart process, also analizyed in* [9]*).*

- *Time inhomogeneous Lévy or Affine processes may be also considered, along the lines of* [20].

4. Example: a multi-currency and functionally symmetric model

The discussion so far focused on the problem of finding models such that the process of the inverted exchange rate is coherent, i.e. belongs to the same class as the original process for the starting FX rate. This represents a first requirement that any FX market model should satisfy. However, when we look at the FX market, more complicated situations may arise. In fact we may not only compute the inverted exchange rate, but we may also construct new exchange rates via products/ratios of exchange rates. The simplest situations we can think of in this sense are given by currency triangles or tetrahedra, see e.g. Figure 2.

The fact that a model is coherent, does not guarantee that products of exchange rates preserve the functional form of the coefficients of the associated SDEs. An example in this sense is given by the basic Heston model, as evidenced by [17].

Even though a model is not fully functionally symmetric, it is possible to use coherent models as building blocks for processes which are stable under suitable multiplications/ratios. Building such models is possible if we change the starting point of the analysis: instead of specifying directly a generic exchange rate as a given state variable, the idea is to consider a family of primitive processes and then construct any exchange rate as a product/ratio of these primitive processes. In the literature, this kind of procedure has been undertaken, in a stochastic volatility setting, by [12]. The idea, which is developed in that paper is inspired by the work of [25], who consider the following model for a generic exchange rate

$$S^{i,j}(t) = \frac{D^i(t)}{D^j(t)}$$

where D^i, D^j are the values of the growth optimal portfolio under currencies i, j. This idea rephrases a classical concept from economics, which is known as the law of one price. Alternatively, the processes D^i, D^j may be thought of as the values w.r.t. currencies i, j of gold, or, using the terminology of [12] a universal numéraire. This is the same principle independently followed by [16] and [17] who terms this approach intrinsic currency valuation framework.

We consider a foreign exchange market in which N currencies are traded and, as in [12] and [25], we start by considering the value of each of these currencies in units of an artificial currency that can be viewed as a universal numéraire. We work under the risk neutral measure defined by the artificial currency and call $S^{0,i}(t)$ the value at time t of one unit of the currency i in terms of our artificial currency (so that $S^{0,i}$ can itself be thought as an exchange rate, between the artificial currency and the currency i). We model each of the $S^{0,i}$ via three main mutually independent stochastic drivers: the first is multi-variate Heston stochastic volatility term [26] with d independent Cox-Ingersoll-Ross (CIR) components [11], $\mathbf{V}(t) \in \mathbb{R}^d$.We further assume that these stochastic volatility components are *common* between the different $S^{0,i}$. This part corresponds to the model of [12]. We generalize the framework by including a time dependent volatility term and jumps. For $0 \leq t \leq T^\star$

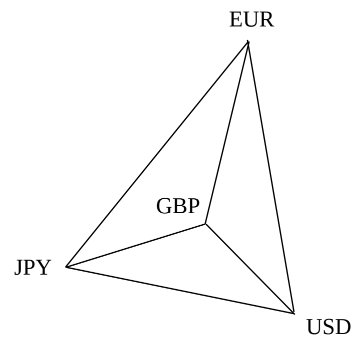

FIGURE 2. A currency tetrahedron

with $T^\star$ a fixed time horizon, we write

$$\begin{aligned}\frac{dS^{0,i}(t)}{S^{0,i}(t^-)} &= (r^0 - r^i)dt - (\mathbf{a}^i)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^0(t) \\ &+ \sum_{l=1}^{f} \Big(-(\sigma_l^i(t))^\top d\mathbf{Z}^l(t) \\ &+ \int_{\mathbb{R}^g} \left(e^{-(\mathbf{b}_l^i)^\top x} - 1 \right) (N_l(dx,dt) - m_l(dx)dt) \Big), \end{aligned} \tag{4.1}$$

$$dV_k(t) = \kappa_k(\theta_k - V_k(t))dt + \xi_k \sqrt{V_k(t)}dW_k(t), \qquad k = 1,\ldots,d; \tag{4.2}$$

where $i = 1, ..., N$ runs over different currencies, $k = 1,\ldots,d$ is the dimension of the stochastic volatility part. The second and the third main driver are a local volatility term introduced via f time dependent e-dimensional time dependent volatility functions $\sigma_l^i(t),\ i = 1,\ldots,N\ l = 1,\ldots,f$ and finally the third introduces jumps, which are included by means of f independent g-dimensional jump processes, with associated random measures N_l and Lévy measures $m_l,\ l = 1,\ldots,f$. The Lévy measures m_l are assumed to satisfy the following condition $\exists\ M > 0$ and $\epsilon > 0$ s.t.

$$\int_{|x|>1} e^{u^\top x} m_l(dx) < \infty, \tag{4.3}$$

$\forall\ u \in [-(1+\epsilon)M, (1+\epsilon)M]^g,\ \forall l = 1, .., f$, which ensures the existence of exponential moments. As far as the stochastic volatility part is concerned, $\kappa_k, \theta_k, \xi_k \in \mathbb{R}$ are parameters in a CIR dynamics, whereas $\sqrt{\mathrm{Diag}(\mathbf{V})}$ denotes the diagonal matrix with the square root of the elements of the vector $\mathbf{V}$ in the main diagonal. This term is multiplied with the linear vector $\mathbf{a}^i \in \mathbb{R}^d$ $(i = 1,\ldots,N)$, in consequence, the dynamics of the exchange rate is also driven by a linear projection of the variance factor $\mathbf{V}$ along a direction parametrized by $\mathbf{a}^i$, so that the total instantaneous variance arising from the stochastic volatility term is $(\mathbf{a}^i)^\top \mathrm{Diag}(\mathbf{V}(t))\mathbf{a}^i dt$. In each monetary area i, the money-market account accrues interest based on the deterministic risk free rate r^i,

$$dB^i(t) = r^i B^i(t)dt, \qquad i = 1,\ldots,N; \tag{4.4}$$

in our universal numéraire analogy r^0 is the artificial currency rate. Finally, in line with [12], we assume an orthogonal correlation structure between the stochastic drivers

$$d\langle Z^0_k, W_h\rangle_t = \rho_k\delta_{kh}dt, \quad k,h = 1,\dots,d, \tag{4.5}$$

together with $d\langle Z^0_k, Z^0_h\rangle_t = \delta_{kh}dt$ and $d\langle W_k, W_h\rangle_t = \delta_{kh}dt$. We notice also that the jumps sizes are multiplied by a $\mathbf{b}^i \in \mathbb{R}^g$, $i = 1,\dots,N$, consequently, the dynamics of the exchange rate is also driven by an f-dimensional family of linear projection of the f-dimensional family of g-dimensional jump processes.

4.1. **Products and ratios of FX rates.** The aim of this section is to show that the general model that we introduced above gives rise to exchange rates which are closed under arbitrary product/ratios. The model describes primitive exchange rates, i.e. exchange rates with respect to an artificial currency, to which an artificial risk-neutral measure $\mathbb{Q}_0$ is associated. Exchange rates among real currencies are constructed by performing two steps: we apply first the Ito formula for semimartingales in order to deduce the dynamics of $S^{i,0}(t) = \left(S^{0,i}(t)\right)^{-1}$, and then we compute the dynamics of $S^{i,j} = S^{i,0}(t)S^{0,j}(t)$ by relying on the product rule. Finally, we show that the resulting process $S^{i,j}$ may be arbitrarily used to construct different exchange rates, meaning that e.g. $S^{i,j} = S^{i,l}S^{l,j}$, such that the coefficients of the SDE preserve their functional form.

Proposition 2. *The dynamics of the exchange rate $S^{i,j} = \left(S^{i,j}(t)\right)_{t\geq 0}$ under the $\mathbb{Q}_0$ risk neutral measure is given by*

$$\begin{aligned}
\frac{dS^{i,j}(t)}{S^{i,j}(t^-)} &= (r^i - r^j)dt \\
&+ \left(\mathbf{a}^i - \mathbf{a}^j\right)^\top \mathrm{Diag}(\mathbf{V}(t))\mathbf{a}^i dt + \left(\mathbf{a}^i - \mathbf{a}^j\right)^\top \mathrm{Diag}(\sqrt{\mathbf{V}(t)})d\mathbf{Z}^0(t) \\
&+ \sum_{l=1}^{f}\Big[(\sigma^i_l(t) - \sigma^j_l(t))^\top(\sigma^i_l(t))dt + (\sigma^i_l(t) - \sigma^j_l(t))^\top d\mathbf{Z}^l(t) \\
&+ \int_{\mathbb{R}}\left(e^{(\mathbf{b}^i_l - \mathbf{b}^j_l)^\top x} - 1\right)\left(N_l(dx,dt) - e^{-(\mathbf{b}^i_l)^\top x}m_l(dx)dt\right)\Big].
\end{aligned} \tag{4.6}$$

Moreover, the dynamics of the exchange rate is invariant under arbitrary products/ratios of exchange rates

Proof. The results directly follows by applying the Ito formula for semimartingales. □

4.2. **The $\mathbb{Q}_i$-risk-neutral process for the FX rate.** So far, the specification of the model has been performed under the risk neutral measure $\mathbb{Q}_0$ associated to the artificial numéraire. The aim of the present section is to present the risk neutral dynamics of the exchange rates together with a precise statement of the relation among the parameters of the model under different measures. This is a key step because a precise understanding of the relatioship among parameters under different measure is necessary e.g. to perform a joint calibration of the model to different volatility surfaces simultaneously, as shown in [12] and [23].

Under the assumptions of the fundamental theorem of asset pricing (cfr. e.g. [4], chapters 13 and 14), investing into the foreign money market account gives a traded asset with value $S^{i,j}B^j/B^i$, and its value has to be a $\mathbb{Q}^i$-martingale. Hence,

$$\begin{aligned}\frac{d\left(\frac{S^{i,j}(t)B^j(t)}{B^i(t)}\right)}{\frac{S^{i,j}(t^-)B^j(t^-)}{B^i(t^-)}} &= \left(\mathbf{a}^i-\mathbf{a}^j\right)^\top \operatorname{Diag}(\sqrt{\mathbf{V}(t)})d(\mathbf{Z}^0)^{\mathbb{Q}_i}(t)\\ &+\sum_{l=1}^{f}\Big[(\sigma_l^i(t)-\sigma_l^j(t))^\top d(\mathbf{Z}^l)^{\mathbb{Q}_i}(t)\\ &+\int_{\mathbb{R}}\left(e^{(\mathbf{b}_l^i-\mathbf{b}_l^j)^\top x}-1\right)\left(N_l(dx,dt)-(m_l)^{\mathbb{Q}_i}(dx)dt\right)\Big].\end{aligned}$$

In the last line we implicitly defined the new Brownian motion vectors $(\mathbf{Z}^0)^{\mathbb{Q}^i}$, $(\mathbf{Z}^l)^{\mathbb{Q}^i}, l=1,...,f$, together with the exponentially tilted Lévy measures $(m_l)^{\mathbb{Q}_i}$, $l=1,..,f$ under the measure $\mathbb{Q}^i$ from the constraint of having a $\mathbb{Q}^i$-local martingale and by Girsanov theorem:

$$d(\mathbf{Z}^0)(t)^{\mathbb{Q}^i}=d\mathbf{Z}^0(t)+\sqrt{\operatorname{Diag}(\mathbf{V}(t))}\mathbf{a}^i dt,\quad i=1,..,N, \tag{4.7}$$

$$d(\mathbf{Z}^l)(t)^{\mathbb{Q}^i}=d\mathbf{Z}^l(t)+\sigma_j^i(t)dt,\quad l=1,...,f\quad i=1,..,N, \tag{4.8}$$

$$(m_l)^{\mathbb{Q}_i}(dx)=e^{-(\mathbf{b}_l^i)^\top x}m_l(dx). \tag{4.9}$$

If we denote by $\mathbb{Q}^0$ the risk neutral measure associated with the universal numéraire, the Radon-Nikodym derivative corresponding to the change of measure from $\mathbb{Q}^0$ to $\mathbb{Q}^i$ reads

$$\begin{aligned}\frac{d\mathbb{Q}^i}{d\mathbb{Q}^0}\bigg|_t &= \exp\Bigg(-\int_0^t\left(\mathbf{a}^i\right)^\top\sqrt{\operatorname{Diag}(\mathbf{V}(s))}d\mathbf{Z}^0(s)-\frac{1}{2}\int_0^t\left(\mathbf{a}^i\right)^\top\operatorname{Diag}(\mathbf{V}(s))\mathbf{a}^i ds\\ &\sum_{l=1}^{f}\Big[-\int_0^t\left(\sigma_l^i(s)\right)^\top d\mathbf{Z}^l(s)-\frac{1}{2}\int_0^t\left(\sigma_l^i(s)\right)^\top\sigma_l^i(s)ds\\ &+\int_0^t\int_{\mathbb{R}^d}\left(e^{-(\mathbf{b}_l^i)^\top x}-1\right)\left(N_l(dx,ds)-m_l(dx)ds\right)\\ &-\int_0^t\int_{\mathbb{R}^d}\left(e^{-(\mathbf{b}_l^i)^\top x}-1+(\mathbf{b}_l^i)^\top x\right)m_l(dx)ds\Big]\Bigg),\end{aligned}$$

hence under the $\mathbb{Q}_i$-risk-neutral measure the exchange rate has the following dynamics

$$\begin{aligned}\frac{dS^{i,j}(t)}{S^{i,j}(t^-)} &= (r^i-r^j)dt+\left(\mathbf{a}^i-\mathbf{a}^j\right)^\top\operatorname{Diag}(\sqrt{\mathbf{V}(t)})d(\mathbf{Z}^0)^{\mathbb{Q}_i}(t)\\ &+\sum_{l=1}^{f}\Big[(\sigma_l^i(t)-\sigma_l^j(t))^\top d(\mathbf{Z}^l)^{\mathbb{Q}_i}(t)\\ &+\int_{\mathbb{R}}\left(e^{(\mathbf{b}_l^i-\mathbf{b}_l^j)^\top x}-1\right)\left(N_l(dx,dt)-(m_l)^{\mathbb{Q}_i}(dx)dt\right)\Big],\end{aligned}$$

as desired.

Given our assumption on the correlation structure in (4.5), we can write the following factorization under $\mathbb{Q}^0$

$$dW_k(t) = \rho_k dZ_k^0(t) + \sqrt{1-\rho_k^2}dZ_k^{\perp}(t), \quad k = 1, ..., d,$$

where $\mathbf{Z}^{\perp}$ is a Brownian motion independent of $\mathbf{Z}^0$. Hence the measure change has also an impact on the variance processes, via the correlations $\rho_k, k = 1, .., d$,

$$dW_k^{\mathbb{Q}^i}(t) = dW_k(t) + \rho_k \left(\mathbf{e}^k\right)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(t))}\mathbf{a}^i dt. \tag{4.10}$$

We finally obtain the dynamics of the instantaneous variance process under the new measure by means of a redefinition of the CIR parameters

$$\begin{aligned}
\rho_k^{\mathbb{Q}^i} =& \rho_k, \\
\kappa_k^{\mathbb{Q}^i} =& \kappa_k + \xi_k \rho_k a_k^i, \\
\theta_k^{\mathbb{Q}^i} =& \theta_k \frac{\kappa_k}{\kappa_k^{\mathbb{Q}^i}},
\end{aligned}$$

so that we can reexpress the variance SDE in its original form

$$dV_k(t) = \kappa_k^{\mathbb{Q}^i}(\theta_k^{\mathbb{Q}^i} - V_k(t))dt + \xi_k \sqrt{V_k(t)}dW_k^{\mathbb{Q}^i}(t). \tag{4.11}$$

The change of measure above is well posed once we can prove that the stochastic exponential in (4.10) is a martingale. It can be easily realized that the Radon-Nikodym derivative may be factorized as a product of independent processes: the first corresponds to the change of measure induced by the multifactor stochastic volatility term, and the second is a product of f mutually independent diffusion and jump terms. By relying on Theorem 2.1 in [37] we can conclude that the stochastic exponential arising from the multi-factor stochastic volatility process is a martingale. We can combine this with the results on time-inhomogeneous Lévy processes in [20] to obtain the claim.

4.3. **Some features.** As we already pointed out, the model we present in this section is intended as an illustration of how our result on the coherency of FX market models may be applied in order to build a fully functional symmetric model for multiple FX rates. Anyhow, we find it interesting to report some interesting features of the approach that we introduced.

A first remark, which is important in view of Figure 1, is that our model presents both a stochastic volatility for the single FX rates and a stochastic correlation among different rates. To see this, we can compute the covariation between two generic FX rates as in [12].

$$\begin{aligned}
& d\left[\int_0^\cdot \frac{dS^{i,j}(s)}{S^{i,j}(s^-)}, \int_0^\cdot \frac{dS^{i,l}(s)}{S^{i,l}(s^-)}\right](t) \\
&= \left(\mathbf{a}^i - \mathbf{a}^j\right)^\top \mathbf{V}(t)\left(\mathbf{a}^i - \mathbf{a}^l\right)dt \\
&+ \sum_{o=1}^{f}\Big[\left(\sigma_o^i(t) - \sigma_o^j(t)\right)^\top\left(\sigma_o^i(t) - \sigma_o^l(t)\right)dt \\
&+ \int_{\mathbb{R}}\left(e^{(\mathbf{b}_o^i - \mathbf{b}_o^j)^\top x} - 1\right)\left(e^{(\mathbf{b}_o^i - \mathbf{b}_o^l)^\top x} - 1\right)N(dx, dt)\Big]
\end{aligned}$$

The result above clearly implies that the variance covariance matrix among exchange rates is a stochastic process, which is a desirable feature given the discussion in the introduction.

A second interesting feature is that the infinitesimal correlation between the logarithm of the FX rate and its infinitesimal variance, which is usually termed skewness, is also a stochastic process. This is a feature arising from the fact that we are considering a multifactor stochastic volatility model. The importance of stochastic skewness is discussed, in the FX setting e.g. in [2].

4.4. **Analytical tractability.** A fundamental feature of the proposed model is the availability of a closed-form solution for the characteristic function of the log-exchange rate, which allows the application of standard Fourier techniques, (see e.g. [32], [34], [38], [7], [31], [19]). For the sake of simplicity, we drop all superscripts indicating the measure and assume that the parameters have been transformed according to the relations illustrated in the Section 4.2. The characteristic function is provided in the following

Proposition 3. *Define $\tau := T - t$. The conditional characteristic function of the log-exchange rate is given by:*

$$\begin{aligned}\varphi_i(u) = \exp\Bigg[& \mathtt{i}ux + \left(r^i - r^j\right)\mathtt{i}u\tau + \sum_{k=1}^{d}\left(A_k^{i,j}(\tau) + B_k^{i,j}(\tau)V_k\right) \\ & \tau\sum_{l=1}^{f}\Bigg(\frac{\mathtt{i}^2u^2\sigma_{l,AV}^2 - \mathtt{i}u\sigma_{l,AV}}{2} \\ & \int_{\mathbb{R}^g}\left(e^{\mathtt{i}u(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1 - \mathtt{i}u(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x\right) \\ & -\mathtt{i}u\left(e^{(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1 - (\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x\right) m_l(dx)\Big]\Bigg)\Bigg],\end{aligned} \tag{4.12}$$

where for $k = 1, .., d$:

$$A_k^{i,j}(\tau) = \frac{\kappa_k\theta_k}{\xi_k^2}\left[\left(Q_k - d_k\right)\tau - 2\log\frac{1 - c_k e^{-d_k\tau}}{1 - c_k}\right],$$

$$B_k^{i,j}(\tau) = \frac{Q_k - d_k}{\xi_k^2}\frac{1 - e^{-d_k\tau}}{1 - c_k e^{-d_k\tau}},$$

$$d_k = \sqrt{Q_k^2 - 4R_kP_k}, \quad c_k = \frac{Q_k - d_k}{Q_k + d_k},$$

$$P_k = \frac{1}{2}\mathtt{i}^2u^2\left(a_k^i - a_k^j\right)^2 - \frac{1}{2}\left(a_k^i - a_k^j\right)^2\mathtt{i}u,$$

$$Q_k = \kappa_k - \mathtt{i}u\left(a_k^i - a_k^j\right)\rho_k\xi_k,$$

$$R_k = \frac{1}{2}\xi_k^2$$

and

$$\sigma_{l,AV}^2 = \frac{1}{T-t}\int_t^T\left(\sigma_l^i(s) - \sigma_l^j(s)\right)^\top\left(\sigma_l^i(s) - \sigma_l^j(s)\right)ds.$$

Proof. The claim directly follows by combining the results found e.g. in [35], [1] and the Lévy-Khintchine formula. □

The article by [12] provides examples of simultaneous calibrations to a triangle of FX implied volatilities of the multifactor stochastic volatility which is nested in the present framework. A similar calibration experiment can be found in [23], where Wishart dynamics are considered. Both articles consider the triangle $EUR-USD-JPY$ for a typical trading day and fit a two-dimensional multifactor stochastic volatility model to the **three** volatility surfaces of $EURUSD$, $USDJPY$ and $JPYEUR$ **simultaneously**. The results, are promising and the reader is referred to these references for a deeper understanding of the fitting procedure and the consequent calibration performance.

5. Conclusions

In this paper we investigated models for FX rates. We observed that we may simultaneously consider an FX rate and its inverse, and we looked for a model class which is coherent, i.e. functionally invariant under inversion of the FX rate while being rich enough in order to accomodate for stylized facts like the presence of volatility smiles in the market.

Our main result shows that affine stochastic volatility models represent the ideal candidates for FX modelling, when it comes to guarantee the requirement above. More generally, however, we may not only consider an FX rate and its inverse, but we may also construct FX rates by means of suitable products or ratios of FX rates. The simplest example in this sense is provided by an FX triangle like EUR-USD-JPY. Using a coherent FX model as a building block, we illustrated a possible way to construct a setting where FX rates are functionally symmetric under such compositions.

The results we presented in this paper are not restricted to the case where a risk-neutral measure exists and may be applied also in the context of the more general benchmark approach, see [3], [24].

Appendix A. Proof of Corollary 1

We follow [13]. The prices of call and put options on S under $\mathbb{Q}_d$ may be expressed, by means of the characteristic functions, via the following formulas:

$$\mathcal{CALL}(S_0, K, r_d, r_f, T) = e^{-r_d T}\left(\frac{1}{2}(F_T - K) + \frac{1}{\pi}\int_0^\infty (F_T f_1 - K f_2)\, du\right) \tag{A.1}$$

$$\mathcal{PUT}(S_0, K, r_d, r_f, T) = e^{-r_d T}\left(\frac{1}{2}(F_T - K) - \frac{1}{\pi}\int_0^\infty (F_T f_1 - K f_2)\, du\right) \tag{A.2}$$

where

$$f_1 = \Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF_T}\right) \tag{A.3}$$

$$f_2 = \Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right) \tag{A.4}$$

Looking now at the foreign domestic parity, we would like to check the agreement between the RHS and the LHS of the following

$$\begin{aligned}&\frac{1}{2}\left(F_T-K\right)+\frac{1}{\pi}\int_0^\infty\left(F_Tf_1-Kf_2\right)d\lambda\\ &=F_TK\left(\frac{1}{2}\left(F_T^{-1}-K^{-1}\right)-\frac{1}{\pi}\int_0^\infty\left(F_T^{-1}f_1'-K^{-1}f_2'\right)d\lambda\right).\end{aligned}\tag{A.5}$$

where f_i' indicates that f_1, f_2 are now computed w.r.t the foreign risk neutral measure $\mathbb{Q}_f$. We substitute the expressions for f_1, f_2, f_1', f_2' so that:

$$\begin{aligned}&F_T-K=\\ &-\frac{1}{\pi}\int_0^\infty\left(F_T\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF_T}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\ &-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}\varphi_f(u-\mathtt{i})}{\mathtt{i}uF_T^{-1}}\right)-F_T\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}\varphi_f(u)}{\mathtt{i}u}\right)\right)du.\end{aligned}\tag{A.6}$$

Now, under Assumption 1, the characteristic functions φ_d, φ_f are related according to Proposition 1 , which allows us to write the following

$$\begin{aligned}&F_T-K=\\ &-\frac{1}{\pi}\int_0^\infty\left(F_T\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF_T}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\ &-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}F_T^{-1}\varphi_d(-u+\mathtt{i}-\mathtt{i})}{\mathtt{i}uF_T^{-1}}\right)\right.\\ &\left.-F_T\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}F_T^{-1}\varphi_d(-u-\mathtt{i})}{\mathtt{i}u}\right)\right)du\\ &=-\frac{1}{\pi}\int_0^\infty\left(F_T\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF_T}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\ &-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u)}{\mathtt{i}u}\right)-F_T\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u-\mathtt{i})}{\mathtt{i}uF_T}\right)\right)du.\end{aligned}\tag{A.7}$$

We regroup terms and obtain:

$$\begin{aligned}&F_T-K=\\ &+\frac{1}{\pi}\int_0^\infty\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u-\mathtt{i})}{\mathtt{i}uF_T}+\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{-\mathtt{i}uF_T}\right)duF_T\\ &-\frac{1}{\pi}\int_0^\infty\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u)}{\mathtt{i}u}+\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{-\mathtt{i}u}\right)duK.\end{aligned}\tag{A.8}$$

We apply the residue Theorem to both integrals and obtain respectively

$$\frac{1}{\pi}\left(\frac{1}{2}2\pi\mathtt{i}\lim_{u\to0}u\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{-\mathtt{i}uF_T}\right)F_T=F_T\tag{A.9}$$

$$\frac{1}{\pi}\left(\frac{1}{2}2\pi\mathtt{i}\lim_{u\to0}u\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{-\mathtt{i}u}\right)K=K\tag{A.10}$$

Thus showing that the foreign domestic parity is indeed satisfied. ■

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