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AN INTRODUCTION TO WHITE NOISE THEORY AND MALLIAVIN CALCULUS FOR FRACTIONAL BROWNIAN MOTION

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Contents

1 Introduction

The stochastic process $B^{(H)}(t)$ which we today call *fractional Brownian motion* (fBm) *with Hurst parameter* $H \in (0,1)$ was originally introduced by Kolmogorov in a study of turbulence. Subsequently many other applications have been suggested. (See Section 3 for more details). For this reason, and also because this 1-parameter family of processes contains the fundamental *classical Brownian motion* as a special case ($H = \frac{1}{2}$), there has been a great increase in the interest and the research activity related to fBm, especially in the last 10 years.

In order to obtain good mathematical models based on fBm it is necessary to have a stochastic calculus for such processes. However, fBm is not a semimartingale (except when $H = \frac{1}{2}$), so the classical stochastic calculus cannot be applied. On the other hand, it turns out that an efficient white noise theory can be constructed and based on this one can introduce stochastic (Wick-Itô) integration and (Malliavin type) differentiation for fBm.

The purpose of this paper is to give an introduction to this newly developed theory. The first part of the paper is mainly a survey of results from [HØUZ], [DHP], [HØ] and [EvdH]. This is the case for most of Sections 2, 3 and 4, except for the fractional Itô formula (Theorem 3.6), which is new in this setting (arbitrary $H \in (0,1)$ and Wick-Itô integration).

In Section 5 we develop the Malliavin calculus for fBm, including a fundamental theorem of fractional stochastic calculus (Theorem 5.3) and integration by parts (Theorem 5.4). These results are new. Then we use these results to give a new proof of the fractional Itô isometry (Theorem 5.6), which was first proved by [EvdH].

Finally in Section 6 we present the multi-dimensional analogues of these results.

2 Classical white noise theory

We begin by recalling the standard setup for the classical white noise probability space. See e.g. [HKPS], [K], [HØUZ] or [AØPU] for more details.

Definition 2.1. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and let $\Omega := \mathcal{S}'(\mathbb{R})$ be its dual, usually called the *space of tempered distributions*. Let μ be the probability measure on the Borel sets $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ defined by the property that

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i < \omega, f >) d\mu(\omega) = \exp(-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}); \quad f \in \mathcal{S}(\mathbb{R}), \tag{1}$$

where $i = \sqrt{-1}$ and $< \omega, f >= \omega(f)$ is the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$.

The measure μ is called the *white noise probability measure*. Its existence follows from the Bochner–Minlos theorem.

Using (1) one can prove that

$$E[< \omega, f >] = 0 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}), \tag{2}$$

where in general

$$E[F(\omega)] = E_\mu[F(\omega)] = \int_\Omega F(\omega) d\mu(\omega)$$

is the expectation of F with respect to μ.

Moreover, we have the isometry

$$E[< \omega, f >^2] = \|f\|^2_{L^2(\mathbb{R})} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}). \tag{3}$$

Based on this we can now define $< \omega, f >$ for an arbitrary $f \in L^2(\mathbb{R})$ as follows:

$$< \omega, f >= \lim_{n\to\infty} < \omega, f_n > \ (\text{limit in } L^2(\mu)), \tag{4}$$

where $f_n \in \mathcal{S}(\mathbb{R})$ is a sequence converging to f in $L^2(\mathbb{R})$. In particular, this makes

$$\tilde{B}(t) := \tilde{B}(t, \omega) :=< \omega, \chi_{[0,t]}(\cdot) > \tag{5}$$

well-defined as an element of $L^2(\mu)$ for all $t \in \mathbb{R}$, where

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \le s \le t \\ -1 & \text{if } t \le s \le 0, \text{ except } t = s = 0 \\ 0 & \text{otherwise} \end{cases}$$

By Kolmogorov's continuity theorem the process $\tilde{B}(t)$ has a continuous version, which we will denote by $B(t)$. It can now be verified that $B(t)$ is a Gaussian process and

$$E[B(t_1)B(t_2)] = \int_{\mathbb{R}} \chi_{[0,t_1]}(s)\chi_{[0,t_2]}(s)ds = \begin{cases} \min(|t_1|,|t_2|) & \text{if } t_1t_2 > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Therefore $B(t)$ is a Brownian motion with respect to the probability law μ.

It follows from (5) that

$$< \omega, f >= \int_{\mathbb{R}} f(t)dB(t) \text{ for all deterministic } f \in L^2(\mathbb{R}). \tag{7}$$

Let $\hat{L}^2(\mathbb{R}^n)$ be the set of all symmetric deterministic functions $f \in L^2(\mathbb{R}^n)$. If $f \in \hat{L}^2(\mathbb{R}^n)$ the *iterated Itô integral* of f is defined by

$$\begin{aligned} I_n(f) &:= \int_{\mathbb{R}^n} f(t)dB^{\otimes n}(t) \\ &:= n! \int_{\mathbb{R}} (\int_{-\infty}^{t_n} \cdots (\int_{-\infty}^{t_2} f(t_1,\ldots,t_n)dB(t_1))dB(t_2)\cdots dB(t_n)). \end{aligned} \tag{8}$$

We now recall the following fundamental result:

Theorem 2.2. (The Wiener-Itô chaos expansion theorem I)

Let $F \in L^2(\mu)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \hat{L}^2(\mathbb{R}^n)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n) \quad (\textit{convergence in } L^2(\mu)). \tag{9}$$

Moreover, we have the isometry

$$E[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}^n)}. \tag{10}$$

By convention we put $I_0(f_0) = f_0$ for constants f_0, and then $\|f_0\|^2 = |f_0|^2$.

In the following we let

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); \quad n = 0,1,2,\ldots \tag{11}$$

denote the *Hermite polynomials* and we let

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}; \quad n = 1, 2, \ldots \tag{12}$$

be the *Hermite functions.* Then $\xi_n \in \mathcal{S}(\mathbb{R})$ and there exist constants C and γ such that

$$|\xi_n(x)| \leq \begin{cases} Cn^{-\frac{1}{12}} & \text{if } |x| \leq 2\sqrt{n} \\ Ce^{-\gamma x^2} & \text{if } |x| > 2\sqrt{n} \end{cases} \tag{13}$$

for all n (See for example, [T], p.26, Lemma 1.5.1.). From [T], $\{\xi_n\}_{n=1}^{\infty}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

Let $\mathcal{J}$ be the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of finite *length* $l(\alpha) = \max\{i; \alpha_i \neq 0\}$, with $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for all i. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J}$ we put $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and we define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(<\omega, \xi_1>)h_{\alpha_2}(<\omega, \xi_2>) \cdots h_{\alpha_n}(<\omega, \xi_n>). \tag{14}$$

Thus, for example,

$$\begin{aligned} \mathcal{H}_{(2,0,3,1)}(\omega) &= h_2(<\omega, \xi_1>)h_0(<\omega, \xi_2>)h_3(<\omega, \xi_3>)h_1(<\omega, \xi_4>) \\ &= (<\omega, \xi_1>^2 - 1)(<\omega, \xi_3>^3 - 3 <\omega, \xi_3>) <\omega, \xi_4>, \end{aligned}$$

since

$$h_0(x) = 1, h_1(x) = x, h_2(x) = x^2 - 1, h_3(x) = x^3 - 3x.$$

Important special cases are the unit vectors

$$\varepsilon^{(k)} = (0, 0, \ldots, 0, 1) \tag{15}$$

with 1 on the k'th entry, 0 otherwise; $k = 1, 2, \ldots$

Note that

$$\mathcal{H}_{\varepsilon^{(k)}}(\omega) = h_1(<\omega, \xi_k>) = <\omega, \xi_k> = \int_{\mathbb{R}} \xi_k(t)dB(t). \tag{16}$$

More generally we have, by a result of Itô[I]:

$$\mathcal{H}_\alpha(\omega) = \int_{\mathbb{R}^{|\alpha|}} \xi^{\hat{\otimes}\alpha}(x)dB^{\otimes|\alpha|}(x) \tag{17}$$

where $\hat{\otimes}$ denotes symmetrized tensor product, i.e. $\xi^{\hat{\otimes}\alpha}(x)$ is the symmetrization with respect to the $l(\alpha)$ variables $x_1, \ldots, x_{l(\alpha)}$ of the tensor product

$$\xi^{\otimes\alpha}(x) := \xi_1^{\otimes\alpha_1}(x_1, \ldots, x_{\alpha_1}) \cdots \xi_m^{\otimes\alpha_m}(x_{l(\alpha)-\alpha_m+1}, \ldots, x_{l(\alpha)}) \tag{18}$$

where $x = (x_1, \ldots, x_{l(\alpha)})$ and $\alpha = (\alpha_1, \ldots, \alpha_{l(\alpha)}) \in \mathcal{J}, \alpha_m \neq 0$.

This is the link between Theorem 2.2 and the following result:

Theorem 2.3. (The Wiener–Itô chaos expansion theorem II)

Let $F \in L^2(\mu)$. Then there exists a unique family $\{c_\alpha\}_{\alpha\in\mathcal{J}}$ of constants $c_\alpha \in \mathbb{R}$ such that

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega) \quad (\textit{convergence in } L^2(\mu)). \tag{19}$$

Moreover, we have the isometry

$$E[F^2] = \sum_{\alpha\in\mathcal{J}} c_\alpha^2 \alpha! \tag{20}$$

We now use Theorem 2.2 and Theorem 2.3 to define the following space $(\mathcal{S})$ of stochastic test functions and the dual space $(\mathcal{S})^*$ of stochastic distributions:

Definition 2.4. a) We define the *Hida space* $(\mathcal{S})$ of stochastic *test functions* to be all $\psi \in L^2(\mu)$ whose expansion

$$\psi(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega)$$

satisfies

$$\sum_{\alpha\in\mathcal{J}} a_\alpha^2 \alpha! (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all } k = 1, 2, \dots \tag{21}$$

where

$$(2\mathbb{N})^\gamma = (2\cdot 1)^{\gamma_1}(2\cdot 2)^{\gamma_2}\cdots(2\cdot m)^{\gamma_m} \quad \text{if } \gamma = (\gamma_1, \dots, \gamma_m) \in \mathcal{J}. \tag{22}$$

b) We define the *Hida space* $(\mathcal{S})^*$ of *stochastic distributions* to be the set of formal expansions

$$G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\sum_{\alpha\in\mathcal{J}} b_\alpha^2 \alpha! (2\mathbb{N})^{-q\alpha} < \infty \quad \text{for some } q < \infty. \tag{23}$$

We equip $(\mathcal{S})$ with the projective topology and $(\mathcal{S})^*$ with the inductive topology. Then $(\mathcal{S})^*$ can be identified with the dual of $(\mathcal{S})$ and the action of $G \in (\mathcal{S})^*$ on $\psi \in (\mathcal{S})$ is given by

$$\langle G, \psi \rangle_{(\mathcal{S})^*,(\mathcal{S})} := \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha b_\alpha \tag{24}$$

In the sequel, we will denote the action $\langle \cdot, \cdot \rangle_{(\mathcal{S})^*,(\mathcal{S})}$ simply with symbol $\langle \cdot, \cdot \rangle$.
In particular, if G belongs to $L^2(\mu) \subset (\mathcal{S})^*$ and $\psi \in (\mathcal{S}) \subset L^2(\mu)$, then

$$\langle G, \psi \rangle = \langle G, \psi \rangle_{L^2(\mu)} = E[G\psi]$$

We can in a natural way define $(\mathcal{S})^*$-valued integrals as follows:

Definition 2.5. Suppose that $Z : \mathbb{R} \longrightarrow (\mathcal{S})^*$ is a given function with property that

$$\langle Z(t), \psi \rangle \in L^1(\mathbb{R}, dt) \quad \forall \psi \in (\mathcal{S}) \tag{25}$$

Then $\int_{\mathbb{R}} Z(t)dt$ is defined to be the unique element of $(\mathcal{S})^*$ such that

$$\langle \int_{\mathbb{R}} Z(t)dt, \psi \rangle = \int_{\mathbb{R}} \langle Z(t), \psi \rangle dt \quad \forall \psi \in (\mathcal{S}) \tag{26}$$

Just as in [HØUZ], Proposition 8.1, one can show that (26) defines $\int_{\mathbb{R}} Z(t)dt$ as an element of $(\mathcal{S})^*$. If (25) holds, we say that $Z(t)$ is *integrable* in $(\mathcal{S})^*$.

Example 2.6. (White noise)
For given $t \in \mathbb{R}$ the random variable $B(t) \in L^2(\mu)$ has the expansion

$$B(t) =< \omega, \chi_{[0,t]}(\cdot) >=< \omega, \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \xi_k(\cdot) >$$

$$= \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds < \omega, \xi_k >= \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds \mathcal{H}_{\varepsilon^{(k)}}(\omega) \tag{27}$$

From this and (13) we see that regarded as a map $B(\cdot) : \mathbb{R} \to (\mathcal{S})^*$, $B(t)$ is differentiable with respect to t and

$$\frac{d}{dt} B(t) = \sum_{k=1}^{\infty} \xi_k(t) \mathcal{H}_{\varepsilon^{(k)}}(\omega) \quad \text{in } (\mathcal{S})^* \tag{28}$$

The expansion on the right hand side of (28) is called *white noise* and denoted by $W(t)$.

The space $(\mathcal{S})^*$ is convenient for the *Wick product*:

Definition 2.7. If $F_i(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(i)} \mathcal{H}_\alpha(\omega); i = 1, 2$, are two elements of $(\mathcal{S})^*$ we define their *Wick product* $(F_1 \diamond F_2)(\omega)$ by

$$(F_1 \diamond F_2)(\omega) = \sum_{\alpha,\beta \in \mathcal{J}} c_\alpha^{(1)} c_\beta^{(2)} \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma \in \mathcal{J}} (\sum_{\alpha+\beta=\gamma} c_\alpha^{(1)} c_\beta^{(2)}) \mathcal{H}_\gamma(\omega) \tag{29}$$

The Wick product is a commutative, associative and distributive (over addition) binary operation on each of the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$.

Example 2.8. i) If F is deterministic then $F \diamond G = F \cdot G$.
ii) If $f \in L^2(\mathbb{R})$ is deterministic, then

$$\int_{\mathbb{R}} f(t)dB(t) = <\omega, f> = \sum_{k=1}^{\infty} (f, \xi_k)_{L^2(\mathbb{R})} <\omega, \xi_k> = \sum_{k=1}^{\infty} (f, \xi_k)_{L^2(\mathbb{R})} \mathcal{H}_{\varepsilon^{(k)}}(\omega).$$

Moreover, if $\|f\|_2 = 1$ then $<\omega, f>^{\diamond n} = h_n(<\omega, f>)$. Hence, if also $g(t) \in L^2(\mathbb{R})$ is deterministic,

$$(\int_{\mathbb{R}} f(t)dB(t)) \diamond (\int_{\mathbb{R}} g(t)dB(t)) = \sum_{i,j=1}^{\infty} (f, \xi_i)_{L^2(\mathbb{R})} (g, \xi_j)_{L^2(\mathbb{R})} \mathcal{H}_{\varepsilon^{(i)}+\varepsilon^{(j)}}(\omega)$$

$$= (\int_{\mathbb{R}} f(t)dB(t)) \cdot (\int_{\mathbb{R}} g(t)dB(t)) - (f, g)_{L^2(\mathbb{R})} \quad \text{(see e.g. [HØUZ])} \tag{30}$$

A fundamental property of the Wick product is the following relation to Itô/Skorohod integration:

Theorem 2.9. *Suppose $f(t, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ is Skorohod integrable. Then $f(t, \cdot) \diamond W(t)$ is dt-integrable in $(\mathcal{S})^*$ and*

$$\int_{\mathbb{R}} f(t, \omega)\delta B(t) = \int_{\mathbb{R}} f(t, \omega) \diamond W(t)dt, \tag{31}$$

where the integral on the left is the Skorohod integral (which coincides with the Itô integral if f is adapted).

Proof. See [HØUZ] for details. □

3 Fractional white noise theory

Recall that if $0 < H < 1$ then the (1-dimensional) *fractional Brownian motion (fBm) with Hurst parameter H* is the Gaussian process $B^{(H)}(t) \in \mathbb{R}$ with mean

$$E[B^{(H)}(t)] = B^{(H)}(0) = 0 \quad \text{for all } t \in \mathbb{R} \tag{32}$$

and covariance

$$E[B^{(H)}(s)B^{(H)}(t)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}; \quad s,t \in \mathbb{R}. \tag{33}$$

Here E denotes the expectation with respect to the probability law of $B^{(H)}(t) = B^{(H)}(t,\omega)$.

Note that if $H = \frac{1}{2}$ then $B^{(H)}(t) = B^{(\frac{1}{2})}(t)$ coincides with the classical Brownian motion. If $\frac{1}{2} < H < 1$ then $B^{(H)}(t)$ is *persistent* while if $0 < H < \frac{1}{2}$ then $B^{(H)}(t)$ is *anti-persistent*. The Hausdorff dimension of the graph of $B^{(H)}(t); t \in [0,1]$ is $2-H$, so the paths of $B^{(H)}(t)$ get smoother as H increases. For $0 < H < \frac{1}{2}$ the quadratic variation of $B^{(H)}(t)$ over $[0,1]$ is infinite, while for $\frac{1}{2} < H < 1$ the quadratic variation is 0.

For these and other results on fBm we refer to [MVN], [S2] and the references therein.

Because of these properties fractional Brownian motion has been suggested as a useful tool in the modeling of physical phenomena and in finance. Kolmogorov, who was the first to study this process, applied it to turbulence ($0 < H < \frac{1}{2}$) (see [S2] for more information). Mandelbrot and Van Ness [MVN] and later Mandelbrot [M] suggested many other applications, particularly in finance. For such applications we refer to [AMN], [BØ], [BHØS], [BSZ], [CQ], [EvdH], [HØ], [HØSa], [HØSu1], [HØSu2], [MST], [Mi], [Ne], [RV], [S1], [Si], [So] and [SV]. In [BSZ] it is argued that temperature can be modeled remarkably well by an Ornstein–Uhlenbeck process driven by an fBm of Hurst parameter $H \approx 0.6 - 0.7$. On the other hand, fBm with Hurst parameter less than $\frac{1}{2}$ appears in the modeling of prices in the Nordic electricity market, according to [Si].

In view of the above it is of interest to develop a stochastic calculus for fBm. If $\frac{1}{2} < H < 1$ the integral with respect to fBm can be defined *pathwise* (with ω as a parameter), as follows:

$$\int_a^b f(t,\omega)\delta B^{(H)}(t) := \lim_{|\Delta t_k| \to 0} \sum_k f(t_k,\omega)(B^{(H)}(t_{k+1}) - B^{(H)}(t_k)). \tag{34}$$

This follows from a general theory of integration with respect to processes of zero quadratic variation, originally due to L.C. Young [Y]. See [N] for more information. However, when applied to finance this type of integration leads to arbitrage. See [D], [DK1], [R] and [S1].

For $\frac{1}{2} < H < 1$ an alternative integration theory based on the Wick product $\diamond$ was introduced by [DHP], as follows:

$$\int_a^b f(t,\omega)dB^{(H)}(t) := \lim_{|\Delta t_k|\to 0} \sum_k f(t_k,\omega) \diamond (B^{(H)}(t_{k+1}) - B^{(H)}(t_k)). \qquad (35)$$

We call these *fractional Itô integrals*, because these integrals share many of the properties of the classical Itô integral. In particular, in contrast to the pathwise integral (34) we have

$$E[\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t)] = 0. \qquad (36)$$

In [HØ] this fractional Itô calculus was extended to a white noise calculus for fBm and applied to finance, still for the case $\frac{1}{2} < H < 1$ only. Then in [EvdH] this theory and its application to finance was extended to be valid for all values of H in $(0,1)$. We now review briefly the approach of [EvdH]:

Fix $H \in (0,1)$.

The main idea is to relate the fractional Brownian motion $B^{(H)}(t)$ with Hurst parameter $H \in (0,1)$ to classical Brownian motion $B(t)$ via the following operator M:

Definition 3.1. The operator $M = M^{(H)}$ is defined on functions $f \in \mathcal{S}(\mathbb{R})$ by

$$\widehat{Mf}(y) = |y|^{\frac{1}{2}-H}\hat{f}(y); \quad y \in \mathbb{R} \qquad (37)$$

where

$$\hat{g}(y) := \int_{\mathbb{R}} e^{-ixy}g(x)dx \qquad (38)$$

denotes the Fourier transform.

This can be restated as follows:

For $0 < H < \frac{1}{2}$ we have

$$Mf(x) = C_H \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}}dt, \qquad (39)$$

where

$$C_H = [2\Gamma(H-\frac{1}{2})\cos(\frac{\pi}{2}(H-\frac{1}{2}))]^{-1}[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2}},$$

with $\Gamma(\cdot)$ denoting the Γ- function.

For $H = \frac{1}{2}$ we have

$$Mf(x) = f(x). \tag{40}$$

For $\frac{1}{2} < H < 1$ we have

$$Mf(x) = C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} dt. \tag{41}$$

The operator M extends in a natural way from $\mathcal{S}(\mathbb{R})$ to the space

$$\begin{aligned} L^2_H(\mathbb{R}) :=& \{f : \mathbb{R} \to \mathbb{R} \text{ (deterministic)}; \ |y|^{\frac{1}{2}-H} \hat{f}(y) \in L^2(\mathbb{R})\} \\ =& \{f : \mathbb{R} \to \mathbb{R}; Mf(x) \in L^2(\mathbb{R})\} \\ =& \{f : \mathbb{R} \to \mathbb{R}; \|f\|_{L^2_H(\mathbb{R})} < \infty\}, \text{ where } \|f\|_{L^2_H(\mathbb{R})} = \|Mf\|_{L^2(\mathbb{R})}. \end{aligned}$$

The inner product on this space is

$$(f, g)_{L^2_H(\mathbb{R})} = (Mf, Mg)_{L^2(\mathbb{R})}. \tag{42}$$

In particular, the indicator function $\chi_{[0,t]}(\cdot)$ is easily seen to belong to this space, for fixed $t \in \mathbb{R}$, and we write

$$M\chi_{[0,t]}(x) = M[0,t](x).$$

Note that if $f, g \in L^2(\mathbb{R}) \cap L^2_H(\mathbb{R})$ then

$$\begin{aligned} &(f, Mg)_{L^2(\mathbb{R})} = (\hat{f}, \widehat{Mg})_{L^2(\mathbb{R})} \\ =& \int_{\mathbb{R}} |y|^{\frac{1}{2}-H} \hat{f}(y)\hat{g}(y)dy = (\widehat{Mf}, \hat{g})_{L^2(\mathbb{R})} = (Mf, g)_{L^2(\mathbb{R})}. \end{aligned} \tag{43}$$

We now define, for $t \in \mathbb{R}$

$$\tilde{B}^{(H)}(t) := \tilde{B}^{(H)}(t, \omega) :=< \omega, M[0,t](\cdot) > \tag{44}$$

Then $\tilde{B}^{(H)}(t)$ is Gaussian, $\tilde{B}^{(H)}(0) = E[\tilde{B}^{(H)}(t)] = 0$ for all $t \in \mathbb{R}$ and, by (3)

$$\begin{aligned} E[\tilde{B}^{(H)}(s)\tilde{B}^{(H)}(t)] =& \int_{\mathbb{R}} M[0,s](x)M[0,t](x)dx \\ =& \int_{\mathbb{R}} \widehat{M[0,s]}(y)\widehat{M[0,t]}(y)dy = \int_{\mathbb{R}} |y|^{1-2H} \widehat{\chi_{[0,s]}}(y)\widehat{\chi_{[0,t]}}(y)dy \\ =& \frac{1}{2}[|t|^{2H} + |s|^{2H} - |s-t|^{2H}] \quad \text{(see [EvdH, (A.10)])} \end{aligned}$$

Therefore, the continuous version $B^{(H)}(t)$ of $\tilde{B}^{(H)}(t)$ is a fractional Brownian motion, as defined in (32)-(33).

Remark. *Note that the underlying probability measure μ is the same as for $B(t)$.*

Let $f(x) = \sum_j a_j \chi_{[t_j, t_{j+1}]}(x)$ be a step function. Then by (44) and linearity

$$
\begin{aligned}
< \omega, Mf > &= \sum_j a_j < \omega, M[t_j, t_{j+1}] > \\
&= \sum_j a_j (B^{(H)}(t_{j+1}) - B^{(H)}(t_j)) = \int_{\mathbb{R}} f(t) dB^{(H)}(t). \qquad (45)
\end{aligned}
$$

Since

$$
\| < \omega, Mf > \|_{L^2(\mu)} = \|Mf\|_{L^2(\mathbb{R})} = \|f\|_{L^2_H(\mathbb{R})},
$$

we see that (45) extends to all $f \in L^2_H(\mathbb{R})$. Comparing with (7) we obtain

$$
\int_{\mathbb{R}} f(t) dB^{(H)}(t) = \int_{\mathbb{R}} Mf(t) dB(t); \quad f \in L^2_H(\mathbb{R}). \qquad (46)
$$

Since $Mf \in L^2(\mathbb{R})$ for all $f \in \mathcal{S}(\mathbb{R})$ we can by (3) define
$M : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ by

$$
< M\omega, f > = < \omega, Mf >; \quad f \in \mathcal{S}(\mathbb{R}), \text{ for } \mu\text{-a.e.} \omega \in \Omega = \mathcal{S}'(\mathbb{R}). \qquad (47)
$$

We now define the following stochastic analogue of $L^2_H(\mathbb{R})$:

$$
L^2_H(\mu) = \{G : \Omega \to \mathbb{R};\ G \circ M \in L^2(\mu)\} \qquad (48)
$$

and

$$
\|G\|_{L^2_H(\mu)} = \|G \circ M\|_{L^2(\mu)}, \qquad (49)
$$

where $(G \circ M)(\omega) = G(M\omega)$ denotes function composition. Note that $L^2_H(\mu) = L^2(\mu \circ M^{-1})$.

Let $\{\xi_k\}_{k=1}^{\infty}$ be the Hermite functions as in Section 2. Define

$$
e_k(x) = M^{-1}\xi_k(x); \quad k = 1, 2, \dots \qquad (50)
$$

Then $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2_H(\mathbb{R})$.

Example 3.2. (Fractional white noise)

From (44) we see that for each t the random variable $B^{(H)}(t,\omega)$ belongs to $L^2(\mu)$, and its chaos expansion (19) can be found as follows:

$$\begin{aligned} B^{(H)}(t) &= <\omega, M[0,t](\cdot)> = < M\omega, \chi_{[0,t]}(\cdot)> \\ &= < M\omega, \sum_{k=1}^{\infty} (\chi_{[0,t]}, e_k)_{L^2_H(\mathbb{R})} e_k(\cdot) > \\ &= < M\omega, \sum_{k=1}^{\infty} (M[0,t], Me_k)_{L^2(\mathbb{R})} e_k(\cdot) > \\ &= \sum_{k=1}^{\infty} (M[0,t], \xi_k)_{L^2(\mathbb{R})} < M\omega, e_k > \\ &= \sum_{k=1}^{\infty} (\chi_{[0,t]}, M\xi_k)_{L^2(\mathbb{R})} < \omega, Me_k > \\ &= \sum_{k=1}^{\infty} \int_0^t M\xi_k(s) ds \mathcal{H}_{\varepsilon^{(k)}}(\omega), \end{aligned} \tag{51}$$

where we have used (44), (47) and (43).

Now define the *fractional white noise* $W^{(H)}(t)$ by the expansion

$$W^{(H)}(t) = \sum_{k=1}^{\infty} M\xi_k(t) \mathcal{H}_{\varepsilon^{(k)}}(\omega). \tag{52}$$

Then it can be shown (see [EvdH]) that $W^{(H)}(t) \in (\mathcal{S})^*$ for all t and

$$\frac{dB^{(H)}(t)}{dt} = W^{(H)}(t) \text{ in } (\mathcal{S})^*. \tag{53}$$

In view of Theorem 2.9 the following definition is natural:

Definition 3.3. (The fractional Itô/Wick integral)

Let $Y : \mathbb{R} \to (\mathcal{S})^*$ be such that $Y(t) \diamond W^{(H)}(t)$ is dt-integrable in $(\mathcal{S})^*$. Then we say that Y is $dB^{(H)}$*-integrable* and we define the integral of $Y(t) = Y(t,\omega)$ with respect to $B^{(H)}(t)$ by

$$\int_{\mathbb{R}} Y(t,\omega) dB^{(H)}(t) = \int_{\mathbb{R}} Y(t) \diamond W^{(H)}(t) dt. \tag{54}$$

Note that by (52) this definition coincides with (46) if $Y = f \in L^2_H(\mathbb{R})$ is deterministic, since in that case, by (43),

$$\int_{\mathbb{R}} f(t) \diamond W^{(H)}(t)dt = \sum_{k=1}^{\infty} (\int_{\mathbb{R}} f(t) M\xi_k(t)dt) \mathcal{H}_{\varepsilon^{(k)}}(\omega)$$
$$= \sum_{k=1}^{\infty} (f, M\xi_k)_{L^2(\mathbb{R})} \mathcal{H}_{\varepsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty} (Mf, \xi_k)_{L^2(\mathbb{R})} \mathcal{H}_{\varepsilon^{(k)}}(\omega)$$
$$= \int_{\mathbb{R}} Mf \diamond W(t)dt = \int_{\mathbb{R}} Mf(t)dB(t).$$

Example 3.4. Using Wick calculus in $(\mathcal{S})^*$ we get

$$\int_0^T B^{(H)}(t)dB^{(H)}(t) = \int_0^T B^{(H)}(t) \diamond W^{(H)}(t)dt$$
$$= \int_0^T B^{(H)}(t) \diamond \frac{dB^{(H)}(t)}{dt} dt = \frac{1}{2}[(B^{(H)}(t))^{\diamond 2}]_0^T = \frac{1}{2}(B^{(H)}(T))^{\diamond 2}$$
$$= \frac{1}{2}(< \omega, M[0,T] >)^{\diamond 2} = \frac{1}{2}[(< \omega, M[0,T] >)^2 - (M[0,T], M[0,T])_{L^2(\mathbb{R})}]$$
$$= \frac{1}{2}(B^{(H)}(T))^2 - \frac{1}{2}\|M[0,T]\|^2_{L^2(\mathbb{R})} = \frac{1}{2}(B^{(H)}(T))^2 - \frac{1}{2}T^{2H}, \tag{55}$$

where we have used (30) and (A9) in [EvdH].

Example 3.5. (The fractional Wick exponential)
Consider the fractional stochastic differential equation

$$dX(t) = \alpha(t)X(t)dt + \beta(t)X(t)dB^{(H)}(t); t \geq 0 \tag{56}$$

which is just a shorthand notation for

$$X(t) = X(0) + \int_0^t \alpha(s)X(s)ds + \int_0^t \beta(s)X(s)dB^{(H)}(s).$$

Here $\alpha(\cdot), \beta(\cdot)$ are locally bounded deterministic functions. To solve this equation we use (53) to rewrite it as a differential equation in $(\mathcal{S})^*$:

$$\frac{dX(t)}{dt} = \alpha(t)X(t) + \beta(t)X(t) \diamond W^{(H)}(t) = X(t) \diamond [\alpha(t) + \beta(t)W^{(H)}(t)]. \tag{57}$$

This is the familiar differential equation for the exponential, but with ordinary product replaced by Wick product. Thus by analogy we guess that the solution is

$$X(t) = X(0) \diamond \exp^{\diamond}(\int_0^t \alpha(s)ds + \int_0^t \beta(s)dB^{(H)}(s)), \tag{58}$$

where

$$\int_0^t \beta(s)dB^{(H)}(s) = \int_{\mathbb{R}} \beta(s)\chi_{[0,t]}(s)dB^{(H)}(s)$$

and, in general,

$$\exp^{\diamond} F = \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}, \text{if convergent in } (\mathcal{S})^*.$$

It can be verified that (58) is indeed the (unique) solution of (57). See [EvdH] or [HØ] for details.

In general we have (see [EvdH, (3.5)] or [HØ, (3.15)])

$$\exp^{\diamond}(< \omega, Mf >) = \exp(< \omega, Mf > -\frac{1}{2}\|Mf\|^2_{L^2(\mathbb{R})}). \tag{59}$$

Therefore the solution can also be written

$$X(t) = X(0) \diamond \exp(\int_0^t \beta(s)dB^{(H)}(s) + \int_0^t \alpha(s)ds - \frac{1}{2}\int_{\mathbb{R}} (M_s\beta(s)\chi_{[0,t]}(s))^2 ds) \tag{60}$$

where M_s is the operator M acting on the variable s.

If $X(0) = x$ is deterministic, this becomes

$$X(t) = x \exp(\int_0^t \beta(s)dB^{(H)}(s) + \int_0^t \alpha(s)ds - \frac{1}{2}\int_{\mathbb{R}} (M_s(\beta(s)\chi_{[0,t]}(s)))^2 ds). \tag{61}$$

In particular, if $\beta(s) = \beta, \alpha(s) = \alpha$ are constants, we get, by using (A 10) in [EvdH],

$$X(t) = x \exp(\beta B^{(H)}(t) + \alpha t - \frac{1}{2}\beta^2 t^{2H});\ t \geq 0. \tag{62}$$

Remark. *Note that if the expansion of the process $Y(s)$ is*

$$Y(s)=\sum_{\alpha\in\mathcal{J}}c_\alpha(s)\mathcal{H}_\alpha(\omega)\quad \textit{for each } s\in\mathbb{R}$$

then the expansion of its $dB^{(H)}$-integral is

$$\int_{\mathbb{R}}Y(s)dB^{(H)}(s)=\int_{\mathbb{R}}(\sum_{\alpha\in\mathcal{J}}c_\alpha(s)\mathcal{H}_\alpha(\omega))\diamond(\sum_{k=1}^{\infty}M\xi_k(s)\mathcal{H}_{\varepsilon^{(k)}}(\omega)ds$$

$$=\int_{\mathbb{R}}\sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}c_\alpha(s)M\xi_k(s)\mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega))ds=\sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}(c_\alpha,M\xi_k)_{L^2(\mathbb{R})}\mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega). \tag{63}$$

In particular, if $\int_{\mathbb{R}}Y(s)dB^{(H)}(s)\in L^2(\mu)$ then

$$E[\int_{\mathbb{R}}Y(s)dB^{(H)}(s)]=0. \tag{64}$$

Indeed, the $dB^{(H)}$-integral shares many of the properties of the Skorohod integral for classical Brownian motion. See section 5.

We end this section by presenting an Ito formula for fractional Brownian motion valid for all H in $(0,1)$. In the case $H>0.5$ such a formula was obtained by Duncan, Hu and Pasik-Duncan [DHP]. A formula for general H similar to ours has been obtained independently by J. van der Hoek [vdH] and C.Bender [B]. Our proof, given below, is different from the proofs of these authors.

Theorem 3.6. (A fractional Itô formula)

Let $f(s,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ belong to $C^{1,2}(\mathbb{R}\times\mathbb{R})$ and assume that the 3 random variables

$$f(t,B^{(H)}(t)),\int_0^t\frac{\partial f}{\partial s}(s,B^{(H)}(s))ds\ \textit{and}\ \int_0^t\frac{\partial^2 f}{\partial x^2}(s,B^{(H)}(s))s^{2H-1}ds \tag{65}$$

all belong to $L^2(\mu)$. Then

$$f(t,B^{(H)}(t))=f(0,0)+\int_0^t\frac{\partial f}{\partial s}(s,B^{(H)}(s))ds+\int_0^t\frac{\partial f}{\partial x}(s,B^{(H)}(s))dB^{(H)}(s)$$

$$+H\int_0^t\frac{\partial^2 f}{\partial x^2}(s,B^{(H)}(s))s^{2H-1}ds. \tag{66}$$

Proof. Let $\alpha \in \mathbb{R}$ be constant and let $\beta : \mathbb{R} \to \mathbb{R}$ be a deterministic differentiable function. Define

$$g(t, x) = \exp(\alpha x + \beta(t)) \tag{67}$$

and put

$$Y(t) = g(t, B^{(H)}(t)). \tag{68}$$

Then

$$\begin{aligned} Y(t) &= \exp(\alpha B^{(H)}(t))\exp(\beta(t)) \\ &= \exp^{\diamond}(\alpha B^{(H)}(t) + \frac{1}{2}\alpha^2 t^{2H})\exp(\beta(t)). \end{aligned} \tag{69}$$

Therefore, by Wick calculus in $(\mathcal{S})^*$,

$$\begin{aligned} \frac{d}{dt}Y(t) &= \exp^{\diamond}(\alpha B^{(H)}(t) + \frac{1}{2}\alpha^2 t^{2H}) \diamond (\alpha W^{(H)}(t) + H\alpha^2 t^{2H-1})\exp(\beta(t)) \\ &\quad + \exp^{\diamond}(\alpha B^{(H)}(t) + \frac{1}{2}\alpha^2 t^{2H})\exp(\beta(t))\beta'(t) \\ &= Y(t)\beta'(t) + Y(t) \diamond (\alpha W^{(H)}(t)) + Y(t)H\alpha^2 t^{2H-1}. \end{aligned}$$

Hence

$$Y(t) = Y(0) + \int_0^t Y(s)\beta'(s)ds + \int_0^t Y(s)\alpha dB^{(H)}(s) + H\int_0^t Y(s)\alpha^2 s^{2H-1}ds.$$

This can be written

$$\begin{aligned} g(t, B^{(H)}(t)) &= g(0,0) + \int_0^t \frac{\partial g}{\partial s}(s, B^{(H)}(s))ds + \int_0^t \frac{\partial g}{\partial x}(s, B^{(H)}(s))dB^{(H)}(s) \\ &\quad + H\int_0^t \frac{\partial^2 g}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds, \end{aligned} \tag{70}$$

which is (66).

Now let $f(t, x)$ be as in Theorem 3.6. Then we can find a sequence $f_n(t, x)$ of linear combinations of functions $g(t, x)$ of the form (67) such that

$$f_n(t, x) \to f(t, x), \frac{\partial f_n}{\partial t}(t, x) \to \frac{\partial f}{\partial t}(t, x), \frac{\partial f_n}{\partial x}(t, x) \to \frac{\partial f}{\partial x}(t, x)$$

and $\frac{\partial^2 f_n}{\partial x^2}(t, x) \to \frac{\partial^2 f}{\partial x^2}(t, x)$ pointwise dominatedly as $n \to \infty$.

By (70) we have for all n

$$f_n(t, B^{(H)}(t)) = f_n(0,0) + \int_0^t \frac{\partial f_n}{\partial s}(s, B^{(H)}(s))ds + \int_0^t \frac{\partial f_n}{\partial x}(s, B^{(H)}(s))dB^{(H)}(s)$$

$$+ H \int_0^t \frac{\partial^2 f_n}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds. \tag{71}$$

Taking the limit of (71) in $L^2(\mu)$ (and hence also in $(\mathcal{S})^*$) we get

$$f(t, B^{(H)}(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds + \lim_{n\to\infty} \int_0^t \frac{\partial f_n}{\partial x}(s, B^{(H)}(s))dB^{(H)}(s)$$

$$+ H \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds. \tag{72}$$

Since $s \to \frac{\partial f}{\partial x}(s, B^{(H)}(s))$ is continuous in $(\mathcal{S})^*$ we have

$$\int_0^t \frac{\partial f_n}{\partial x}(s, B^{(H)}(s))dB^{(H)}(s) = \int_0^t \frac{\partial f_n}{\partial x}(s, B^{(H)}(s)) \diamond W^{(H)}(s)ds$$

$$\to \int_0^t \frac{\partial f}{\partial x}(s, B^{(H)}(s)) \diamond W^{(H)}(s)ds \text{ in } (\mathcal{S})^* \text{ , as } n \to \infty. \tag{73}$$

Comparing (72) and (73) we see that $\int_0^t \frac{\partial f}{\partial x}(s, B^{(H)}(s)) \diamond W^{(H)}(s)ds \in L^2(\mu)$ and (66) follows. □

4 Differentiation

We now recall the approach in [HØ] to differentiation, as modified and extended by [EvdH]:

Definition 4.1. Let $F : \Omega \to \mathbb{R}$ and choose $\gamma \in \Omega$. Then we say F has a *directional M-derivative in the direction* γ if

$$D_\gamma^{(H)} F(\omega) := \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon}[F(\omega + \varepsilon M\gamma) - F(\omega)] \tag{74}$$

exists almost surely in $(\mathcal{S})^*$. In that case we call $D_\gamma^{(H)} F$ *the directional M-derivative* of F in the direction γ.

Example 4.2. (**i**) Suppose $F(\omega) =< \omega, Mf >^n$ for some $f \in L_H^2(\mathbb{R})$, $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\frac{1}{\varepsilon}[F(\omega + \varepsilon M\gamma) - F(\omega)] = \frac{1}{\varepsilon}[< \omega + \varepsilon M\gamma, Mf >^n - < \omega, Mf >^n] \\
&= \frac{1}{\varepsilon}[(< \omega, Mf > +\varepsilon < M\gamma, Mf >)^n - < \omega, Mf >^n] \\
&\to n < \omega, Mf >^{n-1} < M\gamma, Mf > \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Hence we get the *chain rule*

$$D_\gamma^{(H)}(< \omega, Mf >^n) = n < \omega, Mf >^{n-1} < M\gamma, Mf > . \tag{75}$$

In particular, choosing $n = 1$ and $\gamma \in L_H^2(\mathbb{R})$ we get

$$D_\gamma^{(H)}(\int_{\mathbb{R}} f(t) dB^{(H)}(t)) = (M\gamma, Mf)_{L^2(\mathbb{R})} = (\gamma, f)_{L_H^2(\mathbb{R})}. \tag{76}$$

(**ii**) Suppose $G(\omega) =< \omega, Mg >^{\diamond n}$ for some $g \in L_H^2(\mathbb{R})$, $n \in \mathbb{N}$. Assume $\|Mg\|_{L^2(\mathbb{R})} = 1$. Since by Example 2.8 we have $< \omega, Mg >^{\diamond n} = h_n(< \omega, Mg >)$ with h_n as in (11), we get

$$\begin{aligned}
&\frac{1}{\varepsilon}[G(\omega + \varepsilon M\gamma) - G(\omega)] = \frac{1}{\varepsilon}[< \omega + \varepsilon M\gamma, Mg >^{\diamond n} - < \omega, Mg >^{\diamond n}] \\
&= \frac{1}{\varepsilon}[h_n(< \omega + \varepsilon M\gamma, Mg >) - h_n(< \omega, Mg >)] \\
&= \frac{1}{\varepsilon}[h_n(< \omega, Mg > +\varepsilon < M\gamma, Mg >) - h_n(< \omega, Mg >)] \\
&\to h_n'(< \omega, Mg >) < M\gamma, Mg >= n h_{n-1}(< \omega, Mg >) < M\gamma, Mg >,
\end{aligned}$$

by a well-known property of the Hermite polynomials $\{h_n\}_{n=1}^\infty$. Hence the following *Wick chain rule* holds:

$$D_\gamma^{(H)}(< \omega, Mg >^{\diamond n}) = n < \omega, Mg >^{\diamond (n-1)} < M\gamma, Mg > . \tag{77}$$

By linearity this holds also if $\|Mg\|_{L^2(\mathbb{R})} \neq 1$.

Definition 4.3. We say that $F : \Omega \to \mathbb{R}$ is *differentiable* if there exists a function

$$\Psi : \mathbb{R} \to (\mathcal{S})^*$$

such that

$$D_\gamma^{(H)} F(\omega) = \int_{\mathbb{R}} M\Psi(t) M\gamma(t) dt \qquad \text{for all} \quad \gamma \in L_H^2(\mathbb{R}). \tag{78}$$

Then we write

$$D_t^{(H)} F := \frac{\partial^{(H)}}{\partial \omega} F(t, \omega) = \Psi(t) \tag{79}$$

and we call $D_t^{(H)} F$ *the Malliavin derivative* or *the stochastic gradient* of F at t.

Example 4.4. From (75)-(77) we get, for $f \in L_H^2(\mathbb{R})$,

$$D_t^{(H)}(< \omega, Mf >^n) = n < \omega, Mf >^{n-1} f(t) \quad \text{for a.a. } t \tag{80}$$

$$D_t^{(H)}(\int_{\mathbb{R}} f(s) dB^{(H)}(s)) = f(t) \quad \text{for a.a. } t \tag{81}$$

$$D_t^{(H)}(< \omega, Mf >^{\diamond n}) = n < \omega, Mf >^{\diamond(n-1)} f(t) \quad \text{for a.a. } t \tag{82}$$

These examples illustrate that the stochastic gradient satisfies a *chain rule* both with respect to ordinary products and with respect to Wick products. Note that for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{J}$ we have

$$\begin{aligned} \mathcal{H}_\alpha(\omega) &= h_{\alpha_1}(< \omega, \xi_1 >) \cdots h_{\alpha_m}(< \omega, \xi_m >) \\ &=< \omega, \xi_1 >^{\diamond \alpha_1} \cdots < \omega, \xi_m >^{\diamond \alpha_m} \\ &=< \omega, Me_1 >^{\diamond \alpha_1} \cdots < \omega, Me_m >^{\diamond \alpha_m} \end{aligned}$$

Therefore, by (82),

$$\begin{aligned} D_t^{(H)} \mathcal{H}_\alpha(\omega) &= \sum_{i=1}^{\infty} \alpha_i h_{\alpha_{i-1}}(< \omega, Me_i >) \Pi_{j \neq i} h_{\alpha_j}(< \omega, \xi_j >) e_i(t) \\ &= \sum_{i=1}^{\infty} \alpha_i \mathcal{H}_{\alpha - \varepsilon^{(i)}}(\omega) e_i(t). \end{aligned} \tag{83}$$

This motivates the following

Definition 4.5. (Fractional stochastic Sobolev spaces)

Let $\mathbb{D}_{1,2}^{(H)}$ be the set of all $F \in L^2(\mu)$ whose chaos expansion

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega)$$

satisfies

$$\sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha^2 \alpha_i \alpha! \|e_i\|_{L^2(\mathbb{R})}^2 < \infty. \tag{84}$$

If $F \in \mathbb{D}_{1,2}^{(H)}$ we define the *fractional stochastic* derivative of F by

$$D_t^{(H)} F(\omega) = \frac{\partial^{(H)} F}{\partial \omega}(t, \omega) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha \alpha_i \mathcal{H}_{\alpha - \varepsilon^{(i)}}(\omega) e_i(t).$$

Note that if $F \in \mathbb{D}_{1,2}^{(H)}$ then $D_t^{(H)} F(\omega) \in L^2(\lambda \times \mu)$ (where λ denotes Lebesgue measure on $\mathbb{R}$) and

$$\|D_t^{(H)} F\|_{L^2(\lambda \times \mu)}^2 = \sum_{\alpha, i} c_\alpha^2 \alpha_i \alpha! \|e_i\|_{L^2(\mathbb{R})}^2. \tag{85}$$

Next we extend this to $(\mathcal{S})^*$:

Definition 4.6. (The general fractional stochastic gradient)

Let $F \in (\mathcal{S})^*$, with chaos expansion

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega).$$

Then we define the *fractional stochastic gradient* of F by the expansion

$$D_t^{(H)} F(\omega) := \frac{\partial^{(H)} F}{\partial \omega}(t, \omega) := \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha \alpha_i \mathcal{H}_{\alpha - \varepsilon^{(i)}}(\omega) e_i(t) \tag{86}$$

$$= \sum_{\gamma \in \mathcal{J}} [\sum_{\alpha, i: \alpha - \varepsilon^{(i)} = \gamma} c_\alpha \alpha_i e_i(t)] \mathcal{H}_\gamma(\omega) \tag{87}$$

$$= \sum_{\gamma \in \mathcal{J}} [\sum_{i=1}^{l(\gamma)} c_{\gamma + \varepsilon^{(i)}} (\gamma_i + 1) e_i(t)] \mathcal{H}_\gamma(\omega), \tag{88}$$

where $l(\gamma)$ =max$\{i; \gamma_i \neq 0\}$ is the length of γ. One can show that $D_t^{(H)} F \in (\mathcal{S})^*$ for a.a. $t \in \mathbb{R}$.

If $G(t) = G(t, \omega)$ is a time dependent expansion of the form

$$G(t) = \sum_{\alpha \in \mathcal{J}} g_\alpha(t) \mathcal{H}_\alpha(\omega),$$

where $g_\alpha \in L^2_H(\mathbb{R})$ for all $\alpha \in \mathcal{J}$, we define $MG(t)$ by the expansion

$$MG(t) = \sum_{\alpha \in \mathcal{J}} Mg_\alpha(t) \mathcal{H}_\alpha(\omega). \tag{89}$$

In particular, by (52) we see that the relation between fractional and classical white noise is given by

$$W^{(H)}(t) = MW(t). \tag{90}$$

5 Fractional Malliavin calculus

In this section we first study the relations between the fractional and the classical stochastic calculus and then we use this to prove some fundamental results about Malliavin calculus for fractional Brownian motion. As before $D_t^{(H)}$ denotes the Malliavin derivative with respect to $B^{(H)}(\cdot)$. In the classical case ($H = \frac{1}{2}$) we use the notation D_t for the corresponding Malliavin derivative (see Definition 4.6).

Proposition 5.1. (Differentiation)
Let $F \in (\mathcal{S})^*$. Then

$$D_t F = M D_t^{(H)} F \quad \textit{for a.a. } t \in \mathbb{R}. \tag{91}$$

Proof. Let F have the expansion

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega).$$

Then by (88) and (89) we get

$$\begin{aligned} M D_t^{(H)} F &= M\Big(\sum_{\gamma \in \mathcal{J}} \Big[\sum_{i=1}^{l(\gamma)} c_{\gamma+\varepsilon^{(i)}}(\gamma_i + 1) e_i(t)\Big] \mathcal{H}_\gamma(\omega)\Big) \\ &= \sum_{\gamma \in \mathcal{J}} \Big[\sum_{i=1}^{l(\gamma)} c_{\gamma+\varepsilon^{(i)}}(\gamma_i + 1) \xi_i(t)\Big] \mathcal{H}_\gamma(\omega) \\ &= D_t F. \end{aligned}$$

□

Proposition 5.2. (Integration)
Suppose $Y : \mathbb{R} \to (\mathcal{S})^*$ is $dB^{(H)}$-integrable (Definition 3.3). Then

$$\int_{\mathbb{R}} Y(t) dB^{(H)}(t) = \int_{\mathbb{R}} M Y(t) \delta B(t). \tag{92}$$

Proof. Suppose $Y(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) \mathcal{H}_\alpha(\omega)$. Then by (63) and (43) we have

$$\begin{aligned} &\int_{\mathbb{R}} Y(t) dB^{(H)}(t) = \sum_{\alpha \in \mathcal{J}, k \in \mathbb{N}} (c_\alpha, M\xi_k)_{L^2(\mathbb{R})} \mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega) \\ &= \sum_{\alpha \in \mathcal{J}, k \in \mathbb{N}} (M c_\alpha, \xi_k)_{L^2(\mathbb{R})} \mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega) = \int_{\mathbb{R}} M Y(t) \delta B(t). \end{aligned} \tag{93}$$

□

Theorem 5.3. (Fundamental theorem of fractional stochastic calculus)
Suppose $Y:\mathbb{R}\to(\mathcal{S})^*$ is $dB^{(H)}$-integrable. Then

$$D_t^{(H)}(\int_{\mathbb{R}} Y(s)dB^{(H)}(s)) = \int_{\mathbb{R}} D_t^{(H)}Y(s)dB^{(H)}(s) + Y(t) \tag{94}$$

Proof. If $Y(s)=\sum_{\alpha\in\mathcal{J}} c_\alpha(s)\mathcal{H}_\alpha(\omega)$ then by (63) and (86) we get

$$\begin{aligned} D_t^{(H)}(\int_{\mathbb{R}} Y(s)dB^{(H)}(s)) &= D_t^{(H)}(\sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}(c_\alpha, M\xi_k)_{L^2(\mathbb{R})}\mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega)) \\ &= \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}(c_\alpha, M\xi_k)_{L^2(\mathbb{R})}\sum_{i\in\mathbb{N}}(\alpha+\varepsilon^{(k)})_i\mathcal{H}_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}(\omega)e_i(t) \\ &= \sum_{\alpha\in\mathcal{J},k\in\mathbb{N},i\in\mathbb{N}}(c_\alpha, M\xi_k)_{L^2(\mathbb{R})}\alpha_i\mathcal{H}_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}}e_i(t) \\ &+ \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}(c_\alpha, M\xi_k)_{L^2(\mathbb{R})}\mathcal{H}_\alpha(\omega)e_k(t). \end{aligned} \tag{95}$$

Applying (63) to the integrand $D_t^{(H)}Y(s)$ we see that the right hand side of (94) is

$$\begin{aligned} &\sum_{\alpha\in\mathcal{J},k,i\in\mathbb{N}}(c_\alpha, M\xi_k)_{L^2(\mathbb{R})}\alpha_i\mathcal{H}_{\alpha-\varepsilon^{(i)}+\varepsilon^{(k)}}(\omega)e_i(t) \\ &+ \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}}(c_\alpha, e_k)_{L_H^2(\mathbb{R})}e_k(t)\mathcal{H}_\alpha(\omega), \end{aligned}$$

which coincides with (95) since, by (42) and (43),

$$(c_\alpha, e_k)_{L_H^2(\mathbb{R})} = (Mc_\alpha, Me_k)_{L^2(\mathbb{R})} = (Mc_\alpha, \xi_k)_{L^2(\mathbb{R})} = (c_\alpha, M\xi_k)_{L^2(\mathbb{R})}. \tag{96}$$

□

Theorem 5.4. (Fractional integration by parts)
Let $F\in L^2(\mu)$ and $Y:\mathbb{R}\to(\mathcal{S})^*$. Then

$$F\int_{\mathbb{R}} Y(s)dB^{(H)}(s) = \int_{\mathbb{R}} FY(s)dB^{(H)}(s) + \int_{\mathbb{R}} MY(s)MD_s^{(H)}Fds \tag{97}$$

provided that all the terms are well-defined and belong to $L^2(\mu)$.

Proof. The classical ($H = \frac{1}{2}$) integration by parts formula states that

$$F \int_{\mathbb{R}} Y(s)\delta B(s) = \int_{\mathbb{R}} FY(s)\delta B(s) + \int_{\mathbb{R}} Y(s)D_s F ds \tag{98}$$

See e.g. [N, (1.49)].

Combining this with Proposition 5.2 and Proposition 5.1 we get

$$\begin{aligned} F \int_{\mathbb{R}} Y(s) dB^{(H)}(s) &= F \int_{\mathbb{R}} MY(s)\delta B(s) \\ &= \int_{\mathbb{R}} FMY(s)\delta B(s) + \int_{\mathbb{R}} MY(s) D_s F ds \\ &= \int_{\mathbb{R}} M_s(FY(s))\delta B(s) + \int_{\mathbb{R}} MY(s) M_s D_s^{(H)} F ds \\ &= \int_{\mathbb{R}} FY(s) dB^{(H)}(s) + \int_{\mathbb{R}} MY(s) M_s D_s^{(H)} F ds. \end{aligned}$$

□

Since $dB^{(H)}$-integrals have expectation 0 (see (64)) we deduce

Corollary 5.5. *Let $F, Y(s)$ be as in Theorem 5.4. Then*

$$E[F \int_{\mathbb{R}} Y(s) dB^{(H)}(s)] = E[\int_{\mathbb{R}} MY(s) M D_s^{(H)} F ds]. \tag{99}$$

The following fractional Itô isometry was first proved by [EvdH]. We give a different proof, based on the results above:

Theorem 5.6. (The fractional Itô isometry [EvdH])

Let $Y : \mathbb{R} \to (\mathcal{S})^*$. Then

$$\begin{aligned} &E[(\int_{\mathbb{R}} Y(t) dB^{(H)}(t))^2] \\ &= E[\int_{\mathbb{R}} (MY(t))^2 dt] + E[\int_{\mathbb{R}} \int_{\mathbb{R}} D_t^{(H)} M_s^2 Y(s) \cdot D_s^{(H)} M_t^2 Y(t) ds dt] \end{aligned} \tag{100}$$

provided that each of the terms are well-defined. Here M_t indicates that the operator M acts on the variable t, and similarly with M_s.

Proof. By (43), Corollary 5.5, Theorem 5.3 and Corollary 5.5 again we get

$$
\begin{aligned}
&E[\int_{\mathbb{R}}\int_{\mathbb{R}} D_t^{(H)} M_s^2 Y(s) \cdot D_s^{(H)} M_t^2 Y(t) ds dt] \\
&= E[\int_{\mathbb{R}} (\int_{\mathbb{R}} M_s D_t^{(H)} Y(s) \cdot M_s (D_s^{(H)} M_t^2 Y(t)) ds) dt] \\
&= E[\int_{\mathbb{R}} (M_t^2 Y(t) \cdot \int_{\mathbb{R}} D_t^{(H)} Y(s) dB^{(H)}(s) dt] \\
&= E[\int_{\mathbb{R}} M_t^2 Y(t) \cdot \{D_t^{(H)} (\int_{\mathbb{R}} Y(s) dB^{(H)}(s)) - Y(t)\} dt] \\
&= E[\int_{\mathbb{R}} (M_t^2 Y(t) \cdot D_t^{(H)} (\int_{\mathbb{R}} Y(s) dB^{(H)}(s)) dt] - E[\int_{\mathbb{R}} M_t^2 Y(t) \cdot Y(t) dt] \\
&= E[(\int_{\mathbb{R}} Y(s) dB^{(H)}(s))^2] - E[\int_{\mathbb{R}} (M_t Y(t))^2 dt].
\end{aligned}
$$

□

6 The multidimensional case

We now proceed to the multidimensional case. In the following we let $H_1, \dots, H_N$ be N numbers (Hurst coefficients) in $(0,1)$ and we put

$$\mathbf{H} = (H_1, \dots, H_N) \in (0,1)^N.$$

With (Ω, μ) as in Section 2 we let $(\Omega_1, \mu_1), \dots, (\Omega_N, \mu_N)$ be N copies of (Ω, μ) and we put

$$\boldsymbol{\Omega} = (\Omega_1 \times \cdots \times \Omega_N), \quad \boldsymbol{\mu} = (\mu_1 \otimes \cdots \otimes \mu_N).$$

Then the N-dimensional fractional Brownian motion with Hurst vector $\mathbf{H} = (H_1, \dots, H_N)$ is defined by

$$B^{(\mathbf{H})}(t) = (B_1^{(\mathbf{H})}(t), \dots, B_N^{(\mathbf{H})}(t)) \tag{101}$$

where

$$B_k^{(\mathbf{H})}(t) = B_k^{(\mathbf{H})}(t, \omega) = B^{(H_k)}(t, \omega_k); \quad \omega = (\omega_1, \dots, \omega_N) \in \Omega$$

is a 1-dimensional fBm with Hurst coefficient $H_k \in (0,1)$; $k = 1, \dots, N$.

Thus $B^{(\mathbf{H})}(t)$ consists of N independent 1-dimensional fBm's $B^{(H_1)}(t)$,..., $B^{(H_N)}(t)$.

We let $\mathcal{J}^N$ be the set of all N-tuples $\boldsymbol{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(N)})$ with $\alpha^{(j)} = (\alpha_1^{(j)}, \dots, \alpha_{l(\alpha^{(j)})}^{(j)}) \in \mathcal{J}$ for all $j = 1, \dots, N$ and we put

$$\mathcal{H}_{\boldsymbol{\alpha}}(\omega) = \mathcal{H}_{\alpha^{(1)}}(\omega_1) \cdots \mathcal{H}_{\alpha^{(N)}}(\omega_N) \quad \text{for } \boldsymbol{\alpha} \in \mathcal{J}^N. \tag{102}$$

Then the family $\{\mathcal{H}_\alpha\}_{\boldsymbol{\alpha} \in \mathcal{J}^N}$ constitutes an orthogonal basis for $L^2(\boldsymbol{\mu})$ and

$$E_{\boldsymbol{\mu}}[(\mathcal{H}_{\boldsymbol{\alpha}})^2] = \boldsymbol{\alpha}! := \alpha^{(1)}! \cdots \alpha^{(N)}!$$

Therefore every $F \in L^2(\boldsymbol{\mu})$ has a chaos expansion

$$F(\omega) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}^N} c_{\boldsymbol{\alpha}} \mathcal{H}_{\boldsymbol{\alpha}}(\omega) \quad \text{where } c_{\boldsymbol{\alpha}} \in \mathbb{R} \text{ for all } \boldsymbol{\alpha} \in \mathcal{J}^N \tag{103}$$

with

$$\|F\|^2_{L^2(\boldsymbol{\mu})} = \sum_{\boldsymbol{\alpha} \in \mathcal{J}^N} c^2_{\boldsymbol{\alpha}} \boldsymbol{\alpha}!. \tag{104}$$

We can now proceed to define the corresponding stochastic test function spaces $(\mathcal{S})$ and stochastic distribution spaces $(\mathcal{S})^*$, as in Section 2.

Component number n of $B^{(\mathbf{H})}(t)$, $B_n^{(\mathbf{H})}(t)$, has the expansion

$$B_n^{(\mathbf{H})}(t) = B^{(H_n)}(t, \omega_n) = \sum_{k=1}^{\infty} \int_0^t M^{(H_n)} \xi_k(s) ds \mathcal{H}_{\varepsilon^{(k)}}(\omega_n), \tag{105}$$

where $M^{(H_n)}$ is as in Definition 3.1 with $H = H_n$. The corresponding expansion for *component number n of fractional white noise* is

$$W_n^{(\mathbf{H})}(t) = W^{(H_n)}(t, \omega_n) = \sum_{k=1}^{\infty} M^{(H_n)} \xi_k(t) \mathcal{H}_{\varepsilon^{(k)}}(\omega_n). \tag{106}$$

As in Section 3 we have $W_n^{(\mathbf{H})}(t) \in (\mathcal{S})^*$ and

$$W_n^{(\mathbf{H})}(t) = \frac{d}{dt} B_n^{(\mathbf{H})}(t) \quad \text{in } (\mathcal{S})^* \text{ for } n = 1, \ldots, N. \tag{107}$$

(See (51)-(53)).

The Wick product on $(\mathcal{S})^*$ is defined as in Section 2:

$$\sum_{\boldsymbol{\alpha} \in \mathcal{J}^N} a_{\boldsymbol{\alpha}} \mathcal{H}_{\boldsymbol{\alpha}}(\omega) \diamond \sum_{\boldsymbol{\beta} \in \mathcal{J}^N} b_{\boldsymbol{\beta}} \mathcal{H}_{\boldsymbol{\beta}}(\omega) = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{J}^N} a_{\boldsymbol{\alpha}} b_{\boldsymbol{\beta}} \mathcal{H}_{\boldsymbol{\alpha} + \boldsymbol{\beta}}(\omega) \tag{108}$$

Note in particular that if $m, n \in \{1, \ldots, N\}$ with $m \neq n$ then by (102)

$$\mathcal{H}_\alpha(\omega_m) \diamond \mathcal{H}_\beta(\omega_n) = \mathcal{H}_\alpha(\omega_m) \mathcal{H}_\beta(\omega_n). \tag{109}$$

As in Definition 3.3 we now define

Definition 6.1. (The multi-dimensional fractional Itô/Wick integral)

a) If $X : \mathbb{R} \to (\mathcal{S})^*$ is such that $X(t) \diamond W_n^{(\mathbf{H})}(t)$ is dt-integrable in $(\mathcal{S})^*$ then we define

$$\int_{\mathbb{R}} X(t, \omega) dB_n^{(\mathbf{H})}(t) = \int_{\mathbb{R}} X(t, \omega) \diamond W_n^{(\mathbf{H})}(t) dt. \tag{110}$$

b) If $Y : \mathbb{R} \to ((\mathcal{S})^*)^N$ is such that $Y_n(t) \diamond W_n^{(\mathbf{H})}(t)$ is dt-integrable in $(\mathcal{S})^*$ for all $n = 1, \ldots, N$ we say that Y is $dB^{(\mathbf{H})}$-integrable and define

$$\int_{\mathbb{R}} Y(t) dB^{(\mathbf{H})}(t) := \sum_{n=1}^{N} \int_{\mathbb{R}} Y_n(t) dB_n^{(\mathbf{H})}(t). \tag{111}$$

Example 6.2. Let $m \neq n$. Then

$$\int_0^T B_m^{(\mathbf{H})}(t)dB_n^{(\mathbf{H})}(t) = \int_{\mathbb{R}} B_m^{(\mathbf{H})}(T)\chi_{[0,T]}(t) \diamond W_n^{(\mathbf{H})}(t)dt$$

$$= B_m^{(\mathbf{H})}(T) \diamond \int_0^T W_n^{(\mathbf{H})}(t)dt = B_m^{(\mathbf{H})}(T) \diamond B_n^{(\mathbf{H})}(T) = B^{(H_m)}(T,\omega_m)B^{(H_n)}(T,\omega_n)$$

Therefore, if we choose

$$Y_k(t) = \begin{cases} B_m^{(\mathbf{H})}(t)\cdot\chi_{[0,T]}(t) & \text{if k=n} \\ -B_n^{(\mathbf{H})}(t)\cdot\chi_{[0,T]}(t) & \text{if k=m} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int_{\mathbb{R}} Y(t)dB^{(\mathbf{H})}(t) = B^{(H_m)}(T)B^{(H_n)}(T) - B^{(H_n)}(T)B^{(H_m)}(T) = 0,$$

even though $Y \neq 0$.

Proceeding as in Section 4 we are led to the following definition of a stochastic derivative in the direction n: (See Definition 4.5)

Definition 6.3. Let $\mathbb{D}_{1,2}^{(\mathbf{H})}$ be the set of all $F \in L^2(\boldsymbol{\mu})$ whose chaos expansion

$$F(\omega) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}^N} c_{\boldsymbol{\alpha}}\mathcal{H}_{\boldsymbol{\alpha}}(\omega)$$

satisfies

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}^N}\sum_{i=1}^{\infty} c_{\boldsymbol{\alpha}}^2\alpha_i^{(n)}\boldsymbol{\alpha}!\|e_i\|^2 < \infty \quad \text{for } n = 1,\dots,N \tag{112}$$

If $F \in \mathbb{D}_{1,2}^{(\mathbf{H})}$ we define the *fractional stochastic derivative of* F *in direction* n $(n = 1,2,\dots,N)$ by

$$D_{n,t}^{(\mathbf{H})}F = \frac{\partial^{(H_n)}F}{\partial\omega_n}(t,\omega) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}^N}\sum_{i=1}^{\infty} c_{\boldsymbol{\alpha}}\alpha_i^{(n)}\mathcal{H}_{\boldsymbol{\alpha}-\varepsilon^{(n,i)}}(\omega)e_{n,i}(t) \tag{113}$$

where

$$\varepsilon^{(n,i)} = (0,\dots,0,\varepsilon^{(i)},0,\dots,0) \in \mathcal{J}^N$$

with $\varepsilon^{(i)}$ on the n'th place and $e_{n,i}(t) = (M^{(H_n)})^{(-1)}\xi_i(t)$.

We define the *fractional stochastic gradient* of F by

$$\nabla^{(\mathbf{H})}F(t,\omega) = \left(\frac{\partial^{(H_1)}F}{\partial\omega_1}(t,\omega),\dots,\frac{\partial^{(H_N)}F}{\partial\omega_N}(t,\omega)\right). \tag{114}$$

Note that by (112) we have

$$\nabla^{(\mathbf{H})} F(t,\omega) \in L^2(\lambda \times \boldsymbol{\mu}) \quad \text{if } F \in \mathbb{D}_{1,2}^{(\mathbf{H})},$$

where as before λ denotes Lebesgue measure on $\mathbb{R}$.

As in the 1-dimensional case we extend this to $F \in (\mathcal{S})^*$ by setting

$$D_{n,t}^{(\mathbf{H})} F = \frac{\partial^{(\mathbf{H})} F}{\partial \omega_n}(t,\omega) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}^N} \sum_{i=1}^{\infty} c_{\boldsymbol{\alpha}} \alpha_i^{(n)} \mathcal{H}_{\boldsymbol{\alpha} - \varepsilon^{(n,i)}}(\omega) e_i(t). \tag{115}$$

(See Definition 4.6).

We now give the multi-dimensional versions of the results of Section 5. These results can either be obtained similarly to those in Section 5, or by reducing to the 1-dimensional cases, and we therefore omit the proofs.

Proposition 6.4. (Differentiation)

Let $F \in (\mathcal{S})^*$. Then

$$D_{n,t} F = M^{(H_n)} D_{n,t}^{(\mathbf{H})} F \quad \text{for } n = 1, \ldots, N. \tag{116}$$

Proposition 6.5. (Integration)

Suppose $Y : \mathbb{R} \to ((\mathcal{S})^*)^N$ is $dB^{(\mathbf{H})}$-integrable (Definition 6.1). Then

$$\int_{\mathbb{R}} Y(t) dB^{(\mathbf{H})}(t) = \int_{\mathbb{R}} M^{(\mathbf{H})} Y(t) \delta B(t), \tag{117}$$

where

$$M^{(\mathbf{H})} Y(t) = (M^{(H_1)} Y_1(t), \ldots, M^{(H_n)} Y_n(t)). \tag{118}$$

Theorem 6.6. (The fundamental theorem of fractional stochastic calculus)

Suppose $Y : \mathbb{R} \to ((\mathcal{S})^*)^N$ is $dB^{(H)}$-integrable. Then

$$D_{n,t}^{(\mathbf{H})}\Big(\int_{\mathbb{R}} Y(s) dB^{(\mathbf{H})}(s)\Big) = \sum_{j=1}^{N} \int_{\mathbb{R}} D_{n,t}^{(H_n)} Y_j(s) dB_j^{(H)}(s) + Y_n(t). \tag{119}$$

Theorem 6.7. (Fractional integration by parts)

Let $F \in L^2(\mu)$ and $Y : \mathbb{R} \to ((\mathcal{S})^*)^N$. Then

$$F \int_{\mathbb{R}} Y(s) dB^{(\mathbf{H})}(s) = \int_{\mathbb{R}} F Y(s) dB^{(\mathbf{H})}(s) + \int_{\mathbb{R}} M^{(\mathbf{H})} Y(s) \cdot M^{(\mathbf{H})} \nabla^{(\mathbf{H})} F(s) ds, \tag{120}$$

(where the dot $\cdot$ denotes inner product in $\mathbb{R}^N$), provided that all the terms are well defined and belong to $L^2(\boldsymbol{\mu})$.

Theorem 6.8. (The multi-dimensional fractional Itô isometry)
Let $Y : \mathbb{R} \to ((\mathcal{S})^*)^N$. *Then*

$$E[(\int_{\mathbb{R}} Y(t)dB^{(\mathbf{H})}(t))^2] = E[\sum_{n=1}^{N} \int_{\mathbb{R}} (M_t^{(H_n)} Y_n(t))^2 dt]$$

$$+ E[\sum_{m,n=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}} D_{m,t}^{(H_m)}((M_s^{(H_n)})^2 Y_n(s)) \cdot D_{n,s}^{(H_n)}((M_t^{(H_m)})^2 Y_m(t)) ds dt]. \quad (121)$$

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References

[AOPU] K. Aase, B. Øksendal, N- Privault, and J. Ubøe: White noise generalizations of the Clark–Haussmann–Ocone theorem with application to mathematical finance. Finance and Stochastics 4 (2000), 465-496.

[AMN] E.Alòs, O.Mazet, D.Nualart: Stochastic calculus with respect to Gaussian processes. Ann. Probab. 29, (2000), pp. 766- 801.

[B] C. Bender: An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. Manuscript 2002.

[B2] C.Bender: The fractional Itô integral, change of measure and absence of arbitrage. Manuscript 2002.

[BHØS] F. Biagini, Y. Hu, B.Øksendal and A. Sulem: A stochastic maximum principle for processes driven by fractional Brownian motion. Stoch. Proc. and their Appl.100 (2002), pp. 233 - 253.

[BØ] F. Biagini and B. Øksendal: Minimal variance hedging for fractional Brownian motion. Preprint, University of Oslo 1/2002.

[BsZ] D. Brody, J. Syroka and M. Zervos: Dynamical pricing of weather derivatives. Quantitative Finance (to appear).

[CQ] L.Coutin, Z.Qian: Stochastic analysis, rough path analysis and fractional Brownian motions. Prob. Th. and Related Fields 122, (2002) pp. 108-140.

[CKW] N. J. Cutland, P. E. Kopp and W. Willinger: Stock price returns and the Joseph effect: A fractional version of the Black-Scholes model. Seminar on Stochastic Analysis. Random Fields and Applications (Ascona, 1993), 327-351, Progr. Probab., 36, Birkhäuser, Basel, 1995.

[DH] W. Dai and C.C. Heyde: Itô formula with respect to fractional Brownian motion and its application. J. Appl. Math. Stoch. Anal. 9 (1996), 439-448.

[D] A. Dasgupta: Fractional Brownian motion: Its properties and applications to stochastic integration. Ph. D. Thesis, Dept, of Statistics, University of North Carolina at Chapel Hill, 1997.

[DK1] A. Dasgupta and G. Kallianpur: Arbitrage opportunities for a class of Gladyshev processes. Appl. Math. Optim. 41 (2000), 377-385.

[DK2] A. Dasgupta and G. Kallianpur: Chaos decomposition of multiple fractional integrals and applications. Probab. Theory Related Fields 115 (1999), 527-548.

[DHP] T.E. Duncan, Y. Hu and B. Pasik–Duncan: Stochastic calculus for fractional Brownian motion. I. Theory. SIAM J. Control Optim. 38 (2000), 582-612.

[DÜ] L. Decreusefond and A. S. Üstünel: Stochastic analysis of the fractional Brownian motion. Potential Analysis 10 (1999), 177-214.

[EvdH] R. J. Elliott and J. Van der Hoek: A general fractional white noise theory and applications to finance.Mathematical Finance (to appear).

[GN] G. Gripenberg and I. Norros: On the prediction of fractional Brownian motion. J. Appl. Prob. 33 (1996), 400-410.

[HKPS] T. Hida, H. -H Kuo, J. Potthoff and L. Streit: White Noise Analysis. Kluwer, 1993.

[H] Y. Hu: Itô–Wiener chaos expansion with exact residual and correlation, variance inequalities. Journal of Theoretical Probability, 10 (1997), 835-848.

[HØ] Y. Hu and B. Øksendal: Fractional white noise calculus and applications to finance. Preprint, University of Oslo. 10/1999.

[HØSa] Y. Hu, B. Øksendal and D. M. Salopek: Weighted local time for fractional Brownian motion and applications to finance. Preprint, University of Oslo 13/2001.

[HØSu1] Y. Hu, B. Øksendal and A. Sulem: Optimal portfolio in a fractional Black-Scholes market. In S. Albeverio et al. (editors): Mathematical Physics and Stochastic Analysis. World Scientific 2000.

[HØSu2] Y. Hu, B. Øksendal and A. Sulem: Optimal consumption and portfolio in a Black-Scholes market driven by Brownian motion. To appear in Infinite Dimensional Analysis, Quantum Probability and Related Topics.

[HØUZ] H. Holden, B. Øksendal, J. Ubøe and T. Zhang: Stochastic Partial Differential Equations. Birkhauser 1996.

[I] K.Itô: Spectral type of the shift transformation of differential processes with stationary increments. Trans. Amer. Math. Soc. 81 (1956), 253-263.

[J] S. Janson: Gaussian Hilbert spaces. Cambridge University Press, 1997.

[K] H. -H.Kuo: White Noise Distribution Theory. CRC Press 1996.

[Le] A. Le Breton: Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. Stat. Prob. Letters 38 (1998), 263-274.

[Li] S.J. Lin: Stochastic analysis of fractional Brownian motion. Stochastics and Stochastic Reports 55 (1995), 121-140.

[MVN] B.B. Mandelbrot and J.W. Van Ness: Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10 (1968), 422-437.

[M] B.B. Mandelbrot: Fractals and Scaling in Finance: Discontinuity, Concentration, Risk. Springer Verlag, 1997.

[MST] Y.Meyer, F.Sellan, M.S.Taqqu: Wavelets, generalized white noise and fractional integration. The synthesis of fractional Brownian motion. The Journal of Fourier Analysis and Applications 5, (1999) pp. 466-494.

[Mi] Y.Mishura: Fractional stochastic integration and Black-Scholes equation for fractional Brownian model with stochastic volatility. Manuscript, Kiev National Taras Shevchenko University, December 2002.

[Ne] C.Necula: Option pricing in a fractional Brownian motion environment. Manuscript, Academy of Economic Studies, Bucharest, February 2002.

[N] R. Norvaisa: Modelling of stock price changes. A real analysis approach. Finance and Stochastics 4 (2000), 343-369.

[Nu] D. Nualart: The Malliavin Calculus and Related Topics. Springer-Verlag 1995.

[R] L.C.G. Rogers: Arbitrage with fractional Brownian motion. Math. Finance 7 (1997), 95-105.

[RV] F.Russo, P.Vallois: Stochastic calculus with respect to continuous finite quadratic variation processes, Stochastics and Stochastics Reports 70, (2000), pp. 1-40.

[S1] A. Shiryaev: On arbitrage and replication for fractal models: In A. Shiryaev and A. Sulem (editors): Workshop on Mathematical Finance, INRIA, Paris 1998.

[S2] A. Shiryaev: Essentials of Stochastic Finance. World Scientific 1999.

[Si] I. Simonsen: Measuring anti-correlations in the Nordic electricity spot market by wavelets. Manuscript 2002.

[So] T. Sottinen: Fractional Brownian motion, random walks and binary market models. Preprint, University of Helsinki, June 1999.

[SV] T. Sottinen and E. Valkeila: Fractional Brownian motion as a model in finance. Preprint, University of Helsinki 302/2001.

[T] S. Thangavelu: Lectures of Hermite and Laguerre Expansions. Princeton University Press, 1993.

[V] E. Valkeila: On some properties of geometric fractional Brownian motion, Preprint, University of Helsinki, May 1999.

[vdH] J. van der Hoek: Itô type lemma for fractional Brownian motion. Private communication.

[Y] L.C. Young: An inequality of the Hölder type, connected with Stieltjes integration. Acta Math. 67 (1936), 251-282.